GENERALIZATIONS OF THE THEORY OF QUASI-FROBENIUS RINGS.

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CONTENTS.

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ABSTRACT.

In this thesis, we consider several generalizations of the theory of Quasi-Frobenius rings, and construct examples of the classes of rings we introduce. In Chapter 1 we establish well-known results, although the way in which we use idempotents is apparently new.

Chapter 2 is devoted to the study of three generalizations of Quasi-Frobenius rings, namely D-rings, RD-rings (restricted D-rings) and PD-rings (partial D-rings). PD-rings is the largest class of rings we study, and we show that these rings can be considered as a natural generalization of Nakayama's definition of a Quasi-Frobenius ring. D-rings are defined by annihilator conditions, and RD-rings are a generalization of D-rings. We show that RD-rings, hence also D-rings, are semi-perfect, and it follows that they are also PD-rings. We will show that in the self-injective case, these three classes of rings all coincide with a class of rings studied by Osofsky, [30]. We will investigate when the properties described are Morita invariant, and will show that finitely generated modules over D-rings are finite dimensional. Finally, we study group rings over D-rings, RD-rings and PD-rings, and in particular show that if a group ring is a D-ring, then the group is finite and the ring is a D-ring, and further, either the ring is self-injective or the group is Hamiltonian. In Chapter 3 we construct examples of D-rings, RD-rings and PD-rings.

Chapter 4 contains results obtained jointly by Dr. C. R. Hajarnavis and the author. Here, we generalize a result of Hajarnavis, [14], by considering Noetherian rings

each of whose proper homomorphic images are i.p.r.i.-rings. We will obtain a partial structure theory for such rings, and in the prime bounded case show that such rings are Dedekind prime rings.

TERMINOLOGY.

(i) $\mathbb{N}$ denotes the natural numbers, i.e. $\mathbb{N} = \{1,2,3,\dots\}$.

(ii) By a ring R we mean an associative, distributive, but not necessarily commutative ring R with an identity element denoted $1_R \neq 0$, or, when no confusion may arise, simply by 1.

(iii) Whenever appropriate, the absence of the adjective right or left will mean that the condition is (right - left) symmetric. In particular, one-sided ideals of a ring will be referred to specifically as right ideals or left ideals, and the term ideal will only be used to describe a two-sided ideal.

(iv) If A is a subset of a ring R, then we write $A^0 = R$, $A^1 = A$, and inductively we write $A^{n+1} = A^n A$ for each $n \in \mathbb{N}$. If $x \in R$, we write $x^0 = 1_R$, $x^1 = x$, and inductively $x^{n+1} = x^n x$ for each $n \in \mathbb{N}$.

(v) A module M over a ring R will mean either a right or a left unital R-module. If we wish to stress that a module M is a right (respectively, left) R-module, we will write M_R (respectively, ${}_RM$). In particular, R_R (respectively, ${}_RR$) denotes R considered as a right (respectively, left) R-module.

(vi) If R, S are rings and M is a right S- and a left R-module with the actions ${}_RM$ and M_S associating, then we say M is an R-S bimodule, and write ${}_RM_S$.

(vii) If M, N are modules over a ring R, and if $f: M \to N$ is an R-homomorphism, then we write f on the left of its argument; so for each $m \in M$, $f(m) \in N$.

(viii) Module isomorphisms will be denoted by $\cong$, and ring isomorphisms will be denoted by $\simeq$.

(ix) We assume the reader is familiar with:

Zorn's Lemma: If X is a non-empty partially ordered set such that every totally ordered subset of X has an upper bound in X, then X has a maximal element.

(x) We use standard notation throughout. However, we stress that '$\subseteq$' denotes 'contained in', and '$\subset$' denotes 'contained in but not equal to'.

Chapter 1.

ARTINIAN, NOETHERIAN AND SEMI-PERFECT RINGS.

In this chapter we will define Artinian, Noetherian and semi-perfect rings, and will establish some well-known results on these rings. These results will be in constant use in later chapters. The reader may find some of the proofs given here easier than those given in the literature.

We start with some preliminary definitions and results, most of which can be found in [6], [11], [17] and [20].

§ 1 Some basic definitions and results.

1.1.1 Definitions: Let M be a module over a ring R. A submodule N of M is said to be proper if $N \neq M$. A proper submodule N of M is maximal if N is not strictly contained in any other proper submodule of M. A maximal right (respectively left) ideal of R is a maximal submodule of R_R (respectively ${}_RR$). A maximal ideal of R is a proper ideal of R not strictly contained in any other proper ideal of R. Since R has an identity element, Zorn's Lemma ensures that every proper right, left or two-sided ideal of R is contained in a maximal right, left or two-sided ideal of R respectively.

A non-zero submodule N of M is said to be minimal or simple if N does not strictly contain any other non-zero submodule of M. Minimal right, left and two-sided ideals of R are defined in an analogous way to the maximal case.

The socle of M, denoted E(M), is the sum of all simple submodules of M. $E(R_R)$ is called the right socle of R, denoted $E_r(R)$, and $E({}_RR)$ is called the left socle of R, denoted $E_l(R)$.

M is said to be <u>completely reducible</u> if $M = E(M)$. Clearly $E(M)$ is always a completely reducible submodule of M.

1.1.2 <u>LEMMA</u>:

The following conditions on a module M are equivalent:-

(i) M is completely reducible.

(ii) Every submodule of M (including M) is completely reducible.

(iii) Every submodule of M is a direct summand of M.

(iv) M can be expressed as a direct sum of simple submodules.

<u>Proof</u>: See [6], theorem 15.3 .

1.1.3 <u>Definitions</u>: A right or left ideal I of a ring R is said to be <u>nil</u> if for each $x \in I$ there exists $n \in \mathbb{N}$ (depending on x) with $x^n = 0$. I is said to be <u>nilpotent</u> if $I^n = 0$ for some $n \in \mathbb{N}$. The sum of all the nilpotent right ideals of R is called the <u>nilpotent</u> (or <u>Wedderburn</u>) <u>radical</u> of R, denoted $W(R)$. It is well known (see, for example, [11]) that $W(R)$ is also the sum of all nilpotent left ideals of R, and hence is an ideal of R. If $W(R) = 0$, then we say that R is <u>semi-prime</u>. If $W(R)$ is itself nilpotent, then clearly $\frac{R}{W(R)}$ is a semi-prime ring.

The <u>Jacobson radical</u> of a ring R, denoted by $J(R)$, is the intersection of all the maximal right ideals of R. It is well known (see, for example, [17]) that $J(R)$ is also the intersection of all maximal left ideals of R, and hence is an ideal of R, and further $J(R)$ is the unique (right, left or two-sided) ideal of R maximal with respect to the property that $1-j$ has a (right, left or two-sided) inverse in R for each $j \in J(R)$. If $J(R) = 0$, then we say that R is semi-simple. Clearly, $\frac{R}{J(R)}$ is always a semi-simple ring.

A ring R is said to be simple if it has no non-zero proper ideals. Since $J(R)$ is a proper ideal of R, a simple ring is also semi-simple.

Suppose x is an element of a ring R and $x^n = 0$ for some $n \in \mathbb{N}$. Clearly $(1 - x)(1 + x + .. + x^{n-1}) = 1$, so $1 - x$ has a right inverse in R. Hence every nil, and in particular every nilpotent, right ideal of R is contained in $J(R)$. Therefore, $W(R) \subseteq J(R)$. It follows that a semi-simple ring is semi-prime.

The Jacobson radical of a ring has the following useful property, which we shall use frequently.

1.1.4 NAKAYAMA'S LEMMA:

Let M_R be a finitely generated module over a ring R. If $M.J(R) = M$, then $M = 0$.

Proof: See [39], lemma 1.3 .

1.1.5 Notation: When it is clear which ring R is under consideration, we will write E_r, E_l, W and J instead of $E_r(R)$, $E_l(R)$, $W(R)$ and $J(R)$ respectively. The symbols E_r, E_l, W and J will not be used for any other purpose.

§ 2 Artinian rings.

1.2.1 Definitions: A module M over a ring R is Artinian if it has the descending chain condition on R-submodules, i.e. if for any chain $M_1 \supseteq M_2 \supseteq \ldots$ of R-submodules of M, there exists $n \in \mathbb{N}$ (depending on the chain) with $M_n = M_{n+1} = \ldots$. The ring R is right (respectively left) Artinian if R_R (respectively $_RR$) is Artinian. We say R is Artinian if it is both left and right Artinian.

It is well known (see, for example, [11]) that if R is a ring with any of the above chain conditions, then W = J, a nilpotent ideal of R.

1.2.2 THEOREM:

The following conditions on a ring R are equivalent:-

(i) R is semi-simple Artinian.

(ii) R is semi-simple right Artinian.

(iii) R is a (finite) direct sum of simple Artinian rings.

(iv) R_R is completely reducible, i.e. $E_r = R$.

(v) Every right R-module is completely reducible.

Proof: See [6], chapter IV.

Remark: Clearly a finitely generated completely reducible module can be expressed as a direct sum of a finite number of simple submodules. Since any ring R is generated (as a right or a left R-module) by 1_R, the above theorem shows that a semi-simple Artinian ring can be expressed as a direct sum of a finite number of minimal right ideals.

1.2.3 Definitions: A descending chain $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$ of submodules of a module M is said to be a composition series of M of length k if all the factor modules $\frac{M_{i-1}}{M_i}$, $(1 \leq i \leq k)$, are simple. If M has such a series, then any two composition series of M have the same length (see [6], corollary 13.5).

1.2.4 THEOREM:

A ring R is right Artinian if and only if R_R has a composition series.

Proof: See [6], theorem 54.1 .

1.2.5 Definitions: Let X be a subset of a ring R. We define the right annihilator of X to be $\{a \in R: Xa = 0\}$, denoted $r_R(X)$ or simply $r(X)$, and the left annihilator of X to be $\{a \in R: aX = 0\}$, denoted $l_R(X)$ or simply $l(X)$. Clearly $r(X)$ and $l(X)$ are right and left ideals of R respectively, and if X is a right ideal then $r(X)$ is an ideal of R. We will write $rl(X)$ and $lr(X)$ for $r(l(X))$ and $l(r(X))$ respectively. It is easy to see that $X \subseteq rl(X)$ and $X \subseteq lr(X)$, and hence $rlr(X) = r(X)$ and $lrl(X) = l(X)$. If $x \in R$, then we write $r(x)$ and $l(x)$ for $r(\{x\})$ and $l(\{x\})$ respectively.

1.2.6 Definitions: A ring R is said to be a local ring if $\frac{R}{J}$ is a simple Artinian ring, and is said to be a scalar local ring if $\frac{R}{J}$ is a division ring. A local Artinian ring is also called a primary Artinian ring, and a scalar local Artinian ring is also called a completely primary Artinian ring.

If R is a scalar local ring, then clearly J is the unique maximal (right, left or two-sided) ideal of R. It follows that $\{x \in R : x \notin J\}$ is the set of all units of R; that is, J is the set of non-units of R.

1.2.7 Definitions: An element e of a ring R is said to be an idempotent if $0 \neq e = e^2$. Two idempotents e, $f \in R$ are said to be orthogonal if $ef = fe = 0$. More generally, a set $\{e_i\}_{i \in I}$ of idempotents of R is said to be a set of mutually orthogonal idempotents if $e_i e_j = 0$ whenever i, $j \in I$, $i \neq j$. An idempotent $e \in R$ is said to be primitive if it cannot be expressed as the sum of two orthogonal idempotents, and is said to be local if eRe is a scalar local ring. (Notice that e is the identity of the ring eRe).

Let e be an idempotent of a ring R. Then $(1 - e)^2 = 1 - e$, and $eR = r(1 - e)$. Clearly $1 - e$ is not a unit of R, and so $e \notin J$. Thus the Jacobson radical of R contains no idempotents.

1.2.8 LEMMA:

Let e be an idempotent in a ring R, and suppose $I_1,..,I_n$ are non-zero right ideals of R with $eR = I_1 \oplus .. \oplus I_n$. Then there are mutually orthogonal idempotents $e_1,..,e_n$ of R with $e = e_1 + .. + e_n$ and $I_i = e_iR$ for $i = 1,..,n$. Conversely, if $e_1,..,e_n$ are mutually orthogonal idempotents in R, then $e = e_1 + .. + e_n$ is an idempotent, and $eR = e_1R \oplus .. \oplus e_nR$.

Proof: The proof is straightforward and is left to the reader.

1.2.9 Definitions: A non-zero module M over a ring R is said to be

<u>indecomposable</u> if M cannot be expressed as the direct sum of two proper submodules of M. An indecomposable submodule of R_R (respectively $_RR$) is called an <u>indecomposable right</u> (respectively <u>left</u>) <u>ideal</u> of R. An <u>indecomposable ideal</u> of R is an ideal of R that cannot be expressed as the direct sum of two non-zero ideals of R.

1.2.10 <u>LEMMA</u>:

Let e be an idempotent of a ring R. Then the following conditions are equivalent:-

(i) e is primitive.

(ii) eR is indecomposable.

(iii) eRe is a ring with no idempotents other than e ($= 1_{eRe}$).

Further, the following stronger conditions are equivalent:-

(iv) e is local.

(v) eR has a unique maximal right R-submodule.

(vi) $\frac{eR}{eJ}$ is a simple right R-module.

<u>Proof</u>: (i) $\Leftrightarrow$ (ii) follows from 1.2.8, and (i) $\Leftrightarrow$ (iii) is proved as in [6], theorem 54.9 .

(iv) $\Rightarrow$ (v) : Suppose e is local and M_1, M_2 are two proper submodules of eR with $M_1 + M_2 = eR$. Then $e = m_1 + m_2$ for some $m_1 \in M_1$, $m_2 \in M_2$, and so $e = e^2 = m_1e + m_2e$. Now $e \notin eJe$, so for some $j \in \{1,2\}$, $m_je \notin eJe$. But eRe is a scalar local ring, so $e \in eRe = m_jeRe \subseteq M_j$, a contradiction. Clearly now the sum of all proper submodules of eR is the unique maximal submodule of eR.

(v) $\Rightarrow$ (vi) : Suppose M is the unique maximal submodule of eR. Let x be an element of M. Clearly $(1 + x)eR + M = eR$, so $(1 + x)eR = eR$. Therefore, $x = (1 + x)y$ for some $y \in R$. Then $1 = (1 + x)(1 - y)$, so $1 + x$ has a right inverse. Hence $M \subseteq eR \cap J = eJ$. Since $e \notin J$, $M = eJ$, and so $\frac{eR}{eJ}$ is simple.

(vi) $\Rightarrow$ (iv) : Suppose $\frac{eR}{eJ}$ is a simple right R-module, and $x \in eRe$, $x \notin eJe$. Now $xeR + eJ = eR$, so $e = xer + ej$ for some $r \in R$, $j \in J$. Then $e(1 - j) = xer$, and so $e = xer(1 - j)^{-1} = e^2 = xer(1 - j)^{-1}e$. Let $y = er(1 - j)^{-1}e$; clearly $y \in eRe$. Then $e = xy \notin eJe$, so $y \notin eJe$, and in the same way we can find $z \in eRe$ with $yz = e$. Therefore, $z = ez = xyz = xe = x$, so $xy = yx = e$. Hence eRe is a scalar local ring.

Finally we notice that the only idempotent of a scalar local ring is the identity, so (iv) $\Rightarrow$ (iii).

Let I be a non-zero right ideal of a semi-simple Artinian ring R. By 1.1.2 and 1.2.2, I is a direct summand of R, so by 1.2.8, I is generated by an idempotent.

Let e be a primitive idempotent of a semi-simple Artinian ring R. By 1.1.2 and 1.2.2 , eR is completely reducible, and by 1.2.10 , eR is indecomposable. Clearly now eR is a minimal right ideal of R, so by 1.2.10 again, e is a local idempotent.

1.2.11 <u>LEMMA</u>:

Let e be a local idempotent of a ring R, and let M be a simple right R-module. Then $Me \neq 0$ if and only if $M \cong \frac{eR}{eJ}$.

<u>Proof</u>: See [6], theorem 54.12 .

1.2.12 COROLLARY:

Let R be a ring such that $1 = e_1 + .. + e_n$ for some mutually orthogonal local idempotents $e_1,..,e_n$ of R. Then every simple right R-module is isomorphic to $\frac{e_iR}{e_iJ}$ for some i, $1 \leq i \leq n$.

Proof: If M is a simple right R-module, then $0 \neq M = M(e_1 + .. + e_n)$, so $0 \neq Me_i$ for some i. The result now follows from 1.2.11 .

By 1.1.2 and 1.2.8 , and our earlier remarks, a semi-simple Artinian ring satisfies the hypotheses of 1.2.12 , so every simple right module over a semi-simple Artinian ring is isomorphic to a minimal right ideal of R.

Notation: Let A be an ideal of a ring R. If $x, y \in R$ with $x - y \in A$, then we write $x \equiv y \bmod (A)$.

1.2.13 Definitions: Let A be an ideal of a ring R. If $x \in R$ and $x + A$ is an idempotent in $\frac{R}{A}$, (that is, $x \notin A$ and $x \equiv x^2 \bmod (A)$), then we say x is an idempotent modulo A. If, further, there is an idempotent $e \in R$ with $e \equiv x \bmod (A)$, then we say x can be lifted over A. More generally, we say idempotents can be lifted over A if every idempotent modulo A can be lifted over A.

1.2.14 LEMMA:

Let A be a nil ideal of a ring R. Then idempotents can be lifted over A.

Proof: Suppose $x \in R$, $x - x^2 = a \in A$, and $n \in \mathbb{N}$ with $a^n = 0$.

Let $y = x + a(2x - 1) \equiv x \mod (A)$. It is easy to check that $y - y^2 = a^2(4a + 3)$, so clearly $(y - y^2)^m = 0$ for some $m \in \{0\} \cup \mathbb{N}$, $m < n$. The result follows by induction.

1.2.15 LEMMA:

Let I be a right ideal of a ring R, and suppose $I + J = eR + J$ for some idempotent $e \in R$. Then there is an idempotent $f \in I$ such that $Rf = Re$, and $f \equiv e \mod (J)$.

Proof: $e \in I + J$, so $e = x + j$ for some $x \in I$, $j \in J$. Then $e = e^2 = ex + ej$, and so $e(1 - j) = ex$. But $1 - j$ is a unit in R, so $e = ex(1 - j)^{-1} = e^2 = ex(1 - j)^{-1}e$. Let $f = x(1 - j)^{-1}e \in xR \subseteq I$. Now $e = ef$ and $fe = f$, so $f^2 = fef = fe = f$ and $Re = Rf$. Clearly $x \equiv x(1 - j) \mod (J)$, so $x(1 - j)^{-1} \equiv x \mod (J)$. But $e = x + j$, so $e - x = j \in J$ and $x(1 - j)^{-1} \equiv e \mod (J)$. Multiplying on the right by e gives $f \equiv e \mod (J)$ as required.

1.2.16 LEMMA:

Let R be a ring and suppose $a \in R$ is an idempotent modulo J. Then the following conditions are equivalent:-

(i) a can be lifted over J.

(ii) There is an idempotent $f \in aRa$ with $f \equiv a \mod (J)$.

(iii) $aR + J = eR + J$ for some idempotent $e \in R$.

Proof: If $e \equiv a \mod (J)$, $e \in R$, then clearly $eR + J = aR + J$, so (i) $\Rightarrow$ (iii) is trivial. (ii) $\Rightarrow$ (i) is also trivial, so it suffices to show (iii) $\Rightarrow$ (ii). Suppose then e is an idempotent in R and $eR + J = aR + J$. By 1.2.15 we may assume $e \in aR$, $e = ax$ say. Now $a \equiv a^2 \mod (J)$, so $e \equiv ae \mod (J)$, $e - ae = j$ say. Then $e - ae = (e - ae)e = je$, so $(1 - j)e = ae$, and hence $e = (1-j)^{-1}ae$. Let $f = e(1-j)^{-1}a$. $e \in aR$, so $f \in aRa$, and clearly $f = ef$ and $fe = e$, so $f^2 = fef = ef = f$. Now $a \equiv (1-j)a \bmod (J)$, so $(1-j)^{-1}a \equiv a \bmod (J)$, and hence $f = e(1-j)^{-1}a \equiv ea \bmod (J)$. But $a \in eR + J$, so $a \equiv ea \bmod (J)$ and $f \equiv a \bmod (J)$ as required.

1.2.17 COROLLARY:

Let R be a ring, and let $\{a_1, a_2, ..\}$ be a (finite or countably infinite) set of mutually orthogonal idempotents modulo J, such that each a_i can be lifted over J. Then there is a set $\{e_1, e_2, ..\}$ of mutually orthogonal idempotents of R with $e_i \equiv a_i \bmod (J)$ for each i.

Proof: Let e_1 be any idempotent with $e_1 \equiv a_1 \bmod (J)$, and suppose inductively $n \in \mathbb{N}$ and $e_1, .., e_n$ are mutually orthogonal idempotents of R with $e_i \equiv a_i \bmod (J)$ for $1 \leq i \leq n$. Let $f_n = e_1 + .. + e_n$, an idempotent of R by 1.2.8 . Clearly $a_{n+1} \equiv (1 - f_n)a_{n+1}(1 - f_n) \bmod (J)$, so by 1.2.16 there is an idempotent $e_{n+1} \in (1 - f_n)R(1 - f_n)$ with $e_{n+1} \equiv a_{n+1} \bmod (J)$. Clearly now $e_1, .., e_{n+1}$ are mutually orthogonal idempotents, so the result follows by induction.

Remark: It is not always possible to lift an uncountable set of mutually orthogonal idempotents over the Jacobson radical of a ring to a set of mutually orthogonal idempotents, even if (individual) idempotents can be lifted (see [42], example A).

Note: We are now in a position to make some observations about Artinian rings. However, we delay our remarks until we have established a more general situation. The reader may wish to refer to 1.4.4 .

§ 3 Modules

1.3.1 Definitions: A module M over a ring R is said to be finite dimensional if M does not contain an infinite direct sum of non-zero submodules. R is said to be right (respectively left) finite dimensional if R_R (respectively ${}_RR$) is finite dimensional.

A module M over a ring R is said to be uniform if $M \neq 0$ and every submodule of M (including M) is indecomposable. Uniform one-sided ideals of R are defined in the obvious way.

Clearly any non-zero submodule of a finite dimensional (respectively uniform) module is also finite dimensional (respectively uniform).

1.3.2 Definitions: Let N be a submodule of a module M over a ring R. N is said to be essential in M if for any non-zero submodule N' of M, $N \cap N' \neq 0$. A submodule L of M containing N is said to be an essential extension of N in M if N is essential in L. N is said to be superfluous in M if for any proper submodule N' of M, $N + N' \neq M$. Essential and superfluous one-sided ideals of R are defined in the obvious way.

Let I be a right ideal of a ring R. If $I \neq R$, then I is contained in a maximal right ideal of R, so $I + J \neq R$. Therefore, J is superfluous in R_R. Suppose $I \not\subseteq J$, so I is not contained in some maximal right ideal M of R. Then $I + M = R$, so I is not superfluous. Clearly now I is superfluous in R if and only if $I \subseteq J$.

1.3.3 THEOREM:

Let N be a submodule of a finite dimensional module M over a ring R. Then

(i) If $N \neq 0$, then N contains a uniform submodule.

(ii) There are uniform submodules $U_1, .., U_n$ of M with $N \oplus U_1 \oplus .. \oplus U_n$ essential in M. Further, if $V_1, .., V_m$ are non-zero submodules of M with $N \oplus V_1 \oplus .. \oplus V_m$ essential in M, then $m \leqslant n$, and $m = n$ if and only if each V_i is uniform.

Proof: See [13], theorem 1.07 .

1.3.4 Definitions: Let M be a finite dimensional module over a ring R. Putting $N = 0$ in theorem 1.3.3(ii), we see that there is a unique $n \in \mathbb{N} \cup \{0\}$ such that M contains a direct sum of n uniform submodules which is essential in M, and any direct sum of non-zero submodules of M has at most n terms. We may therefore define the dimension of M, denoted dim (M), to be this integer n. If R is right (respectively left) finite dimensional, then the right (respectively left) ideal dimension of R is defined to be dim (R_R) (respectively dim $({}_R R)$).

Let M be a finite dimensional module. Clearly

(i) dim $(M) = 0$ if and only if $M = 0$.

(ii) dim $(M) = 1$ if and only if M is uniform.

(iii) If M is completely reducible, then dim $(M) = n$ if and only if M can be expressed as the direct sum of n simple submodules of M.

From (iii) above, and from 1.2.2 , 1.2.8 and 1.2.10 we see that a semi-simple Artinian ring R is right and left finite dimensional, and that the right ideal dimension and the left ideal dimension of R are equal, and is n ($\in \mathbb{N}$) if and only if there are n mutually orthogonal primitive idempotents of R whose sum is 1_R.

1.3.5 Definitions: Let R be a ring. A *sequence* of R-modules $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow \cdots$ consists of a (finite or countably infinite) ordered family of R-modules $M_1, M_2, M_3, \ldots$ with R-homomorphisms $f_1: M_1 \longrightarrow M_2$, $f_2: M_2 \longrightarrow M_3$, ... We say such a sequence is *exact* if $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$ for each i. In particular, an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a *short exact sequence*, and such a sequence is said to *split* if $\operatorname{Im} f$ (= $\operatorname{Ker} g$) is a direct summand of M.

Clearly a sequence $0 \longrightarrow L \xrightarrow{f} M$ is exact if and only if f is a monomorphism, and a sequence $M \xrightarrow{g} N \longrightarrow 0$ is exact if and only if g is an epimorphism.

1.3.6 Definition: A module P over a ring R is said to be *projective* if given any exact sequence $A \xrightarrow{f} B \longrightarrow 0$ and any R-homomorphism $g: P \longrightarrow B$, there is an R-homomorphism $h: P \longrightarrow A$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "g"] & \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

commutes, i.e. such that $fh = g$.

It is straightforward to check that if R is a ring, then ${}_R R$ and R_R are projective modules.

1.3.7 LEMMA:

Let P be a module over a ring R. Then P is projective if and only if every short exaet sequence

$$0 \to A \to B \to P \to 0$$

splits.

Proof: See [6], 56.3 .

1.3.8 THEOREM:

Let R be a ring and let $\{P_i\}_{i\in I}$ be a set of right R-modules. Then the direct sum $\sum_{i\in I} \oplus P_i$ is projective if and only if each P_i, $i \in I$, is projective.

Proof: See [6], theorem 56.5 .

There is a fairly obvious dual concept to the notion of a projective module, namely:-

1.3.9 Definitions: A module M over a ring R is said to be injective if given any exact sequence $0 \to A \xrightarrow{f} B$ and any R-homomorphism $g:A \to M$, there is an R-homomorphism $h:B \to M$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

commutes, i.e. such that $hf = g$. R is said to be right (respectively left) self-injective if R_R (respectively $_RR$) is injective, and is said to be self-injective if it is both right and left self-injective.

Definition: Let I be a right ideal of a ring R, and let M be a right R-module. An R-homomorphism $f:I \to M$ is said to be given by left multiplication (by an element of M) if there is an element $m \in M$ such that $f(x) = mx$ for each $x \in I$.

1.3.10 THEOREM:

Let R be a ring, and let M be a right R-module. Then the following conditions are equivalent:-

(i) M is injective.

(ii) Every short exact sequence $0 \to M \to A \to B \to 0$ splits.

(iii) Every R-homomorphism from a right ideal of R to M is given by left multiplication.

Proof: See [6], theorems 57.9 and 57.14 .

Putting M = R in 1.3.10(iii) immediately gives

1.3.11 COROLLARY:

A ring R is right self-injective if and only if every R-homomorphism from a right ideal of R to R is given by left multiplication by an element of R.

1.3.12 Remark: It is easy to see that if I is a right ideal of a right self-injective ring R, and if I_o is a maximal essential extension of I in R_R (such exists by Zorn's Lemma), then I_o is a direct summand of R, so $I_o = eR$ for some $e = e^2 \in R$. Hence every non-zero right ideal of R has an idempotently generated essential extension.

§ 4 Semi-perfect rings.

1.4.1 Definition: Let M be a module over a ring R. A projective cover of M is a projective R-module P together with an R-epimorphism $f:P \longrightarrow M$ such that Ker f is superfluous in P.

The Eckmann - Schöpf theorem (see [6] theorem 57.13) shows that any module can be embedded (in a minimal way) in an injective module, called its injective hull. As observed by Bass in [3], if we consider projective modules as a dual concept to that of injective modules, then projective covers are the dual notion to injective hulls. However, in contrast to the Eckmann - Schöpf theorem, projective covers seldom exist; for example, an abelian group (considered as a module over $\mathbb{Z}$, the ring of rational integers) has a projective cover only if it is free.

1.4.2 Definitions: A ring R is said to be right (respectively left) semi-perfect if every simple right (respectively left) R-module has a projective cover, and is said to be right (respectively left) perfect if every right (respectively left) R-module has a projective cover. R is said to be semi-perfect (respectively perfect) if it is both left and right semi-perfect (respectively perfect).

Semi-perfect and perfect rings were first studied by Bass in [3]. The following important result can be found in his paper. The proof given here, while essentially the same as Bass's proof, is shorter than the proof given in [3].

1.4.3 <u>THEOREM:</u>

The following conditions on a ring R are equivalent:-

(i) R is right semi-perfect.

(ii) R is left semi-perfect.

(iii) $\frac{R}{J}$ is semi-simple Artinian and idempotents can be lifted over J.

(iv) Every finitely generated right R-module has a projective cover.

(v) Every finitely generated left R-module has a projective cover.

<u>Proof</u>: Since (iii) is right - left symmetric, and (iv) $\Rightarrow$ (i) is trivial, it suffices to prove (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iii):- Let M be a maximal right ideal of R, and suppose $f:P \rightarrow \frac{R}{M}$ is a projective cover of $\frac{R}{M}$. Choose $p \in P$ with $f(p) = 1 + M$. Now $f(pR) = \frac{R}{M}$, so $P = pR + \text{Ker } f$. But Ker f is superfluous in P, so $P = pR$. Let $r(p) = \{x \in R : px = 0\}$. Now the natural sequence $0 \rightarrow r(p) \rightarrow R \rightarrow pR \rightarrow 0$ is exact, so by 1.3.7 there is a right ideal I of R with $I \oplus r(p) = R$. Clearly $f(p.r(p)) = f(0) = 0$, and since $f(p) = 1 + M$, $r(p) \subseteq M$. Therefore, $M = (I \cap M) \oplus r(p)$. Suppose K is a right ideal of R and $(I \cap M) + K = R$. Then $I = (I \cap M) + (I \cap K)$, so $pR = p(I \oplus r(p)) = pI = p(I \cap M) + p(I \cap K) \subseteq \text{Ker } f + p(I \cap K)$. But Ker f is superfluous in pR, so $pR = p(I \cap K)$, whence $(I \cap K) + r(p) = R$. But $I \oplus r(p) = R$, so clearly $I \cap K = I$, i.e. $I \subseteq K$. Thus $K = K + (I \cap M) = R$. Therefore $I \cap M$ is a superfluous right ideal, so $I \cap M \subseteq J$. Now $R = I \oplus r(p)$, so by 1.2.8, $r(p) = eR$ for some $e = e^2 \in R$. Therefore $M = r(p) \oplus (I \cap M) = eR + J$. Suppose $E_r(\frac{R}{J}) \neq \frac{R}{J}$. Then there is a maximal right ideal M with $E_r(\frac{R}{J}) \subseteq \frac{M}{J}$. But by the above, $M = eR + J$ for some $e = e^2 \in R$, and

$\frac{R}{M} = \frac{R}{eR + J} \cong \frac{(1 - e)R + J}{J}$, a simple module, so

$\frac{(1 - e)R + J}{J} \subseteq E_r(\frac{R}{J}) \subseteq \frac{M}{J}$, whence $(1 - e)R + J \subseteq M = eR + J \neq R$, a contradiction. Hence $E_r(\frac{R}{J}) = \frac{R}{J}$, so by 1.2.2, $\frac{R}{J}$ is semi-simple Artinian. Suppose $a \in R$ is a primitive, hence local, idempotent modulo J. Clearly $(1 - a)R + J$ is a maximal right ideal of R, so $(1 - a)R + J = eR + J$ for some $e = e^2 \in R$. By 1.2.16, $1 - a \equiv f \bmod (J)$ for some $f = f^2 \in R$, so $a \equiv 1 - f \bmod (J)$, and $1 - f$ is an idempotent in R. Hence primitive idempotents modulo J can be lifted over J. Finally, since $\frac{R}{J}$ is semi-simple Artinian, any idempotent modulo J can be expressed as a finite sum of mutually orthogonal primitive idempotents modulo J, so it follows from the above and 1.2.17 that any idempotent can be lifted over J.

(iii) $\Longrightarrow$ (iv):- Let M be a finitely generated right R-module. $\frac{M}{MJ}$ is a right $\frac{R}{J}$ - module, so by 1.2.2 $\frac{M}{MJ}$ is completely reducible as an $\frac{R}{J}$ - module, and hence also as an R-module. By 1.1.2 and 1.2.12, there is an indexing set I, local idempotents $e_i \in R$ for each $i \in I$, and an isomorphism $f: \frac{M}{MJ} \to \sum_{i \in I} \oplus \frac{e_iR}{e_iJ}$. Let $P = \sum_{i \in I} \oplus\, e_iR$. By 1.3.8, each e_iR is projective, so P is projective. Let $g: P \to \sum_{i \in I} \oplus \frac{e_iR}{e_iJ}$ and $h: M \to \frac{M}{MJ}$ be the canonical epimorphisms. P is projective, so there is an R-homomorphism $t: P \to M$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "t"'] \arrow[d, "f^{-1}g"] \\ M \arrow[r, "h"] & \frac{M}{MJ} \end{tikzcd}$$

commutes. Now $ht(P) = f^{-1}g(P) = \frac{M}{MJ}$, so $t(P) + MJ = M$. But now $(\frac{M}{t(P)})J = \frac{M}{t(P)}$, and M is finitely generated, so by Nakayama's Lemma, $\frac{M}{t(P)} = 0$, i.e. $M = t(P)$.

Thus t is an R-epimorphism. Suppose Q is a submodule of P and $Q + \text{Ker } t = P$. Now $0 = ht(\text{Ker } t) = f^{-1}g(\text{Ker } t)$, so $\text{Ker } t \subseteq \text{Ker } g = PJ$. Thus $Q + PJ = P$, so $(\frac{P}{Q})J = \frac{P}{Q}$. But M, and hence $\frac{M}{MJ}$, is finitely generated, so I is a finite indexing set, whence P is finitely generated. Therefore, by Nakayama's Lemma, $\frac{P}{Q} = 0$, i.e. $P = Q$. Thus Ker t is superfluous in P, so t is a projective cover of M as required.

We note that if R is a right or left Artinian ring, then J $(= W)$ is nilpotent, so by 1.2.14 and the above theorem, R is semi-perfect.

1.4.4 Remarks: Let R be a semi-perfect ring.

(i) Suppose x is a primitive, hence local, idempotent modulo J. There is an idempotent $e \in R$ with $e \equiv x \bmod (J)$, so $eR + J = xR + J$. But now $\frac{eR}{eJ} \cong \frac{eR + J}{J} = \frac{xR + J}{J}$ is a simple right R-module, so e is a local idempotent of R.

(ii) Suppose e is a primitive idempotent of R, so by 1.2.10 eRe contains no idempotents other than e. But now by 1.2.16, if $a \in eRe$ is an idempotent modulo J, then $e \equiv a \bmod (J)$, so $(e + J)\frac{R}{J}(e + J)$ $(\subseteq \frac{R}{J})$ contains no idempotents other than $e + J$, so e is primitive modulo J. Hence (by (i)) e is a local idempotent of R.

(iii) It is now easy to see that $1 = e_1 + .. + e_n$ for some mutually orthogonal local idempotents $e_1, .. e_n$ of R, where n is the right (or left) ideal dimension of $\frac{R}{J}$.

(iv) If I is a right ideal of R, $I \not\subseteq J$, then $\frac{I + J}{J}$ is generated by an idempotent of $\frac{R}{J}$, so $I + J = eR + J$ for some idempotent $e \in R$. By 1.2.15, we may assume $e \in I$, so clearly $I = eR + (I \cap J) = eR \oplus (1 - e)(I \cap J)$. It follows that a right ideal of R is contained in J if and only if it contains no idempotents.

1.4.5 THEOREM:

The following conditions on a ring R are equivalent:-

(i) R is right perfect.

(ii) R satisfies the descending chain condition on principal left ideals.

(iii) $\frac{R}{J}$ is Artinian and every non-zero left R-module has non-zero socle.

(iv) $\frac{R}{J}$ is Artinian and every non-zero right R-module has a maximal submodule.

Proof: [3], theorem P, gives (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), and [39], theorem 2.3, gives (i) $\Leftrightarrow$ (iv).

§ 5 Idempotents.

We have already seen how useful idempotents can be in studying Artinian and semi-perfect rings, and in this section we will establish some more useful results on idempotents. We start this section by considering when two idempotently generated right ideals are isomorphic.

The following theorem is well known in the semi-perfect case. The second part of this theorem was first proved (using composition series arguments) for Artinian rings by Dwan in [8]. Our generalization does not seem to occur in the literature.

1.5.1 THEOREM:

Let e, f be idempotents in a ring R. Then the following conditions are equivalent:-

(i) $eR \cong fR$

(ii) $Re \cong Rf$

(iii) $\frac{eR}{eJ} \cong \frac{fR}{fJ}$

(iv) $\frac{Re}{Je} \cong \frac{Rf}{Jf}$

(v) There are elements $u \in eRf$, $v \in fRe$ with $uv = e$, $vu = f$.

(vi) There are elements $u, v \in R$ with $uv = e$, $vu = f$.

Further, if e, f are local idempotents, then the above conditions are also equivalent to:-

(vii) $eRf \not\subseteq J$.

(viii) $fRe \not\subseteq J$.

Proof: Suppose $\varphi: eR \rightarrow fR$ is an isomorphism. Let $v = \varphi(e) = \varphi(e^2) = \varphi(e)e \in fRe$ and $u = \varphi^{-1}(f) \in eRf$. Now $uv = \varphi^{-1}(f\varphi(e)) = \varphi^{-1}(\varphi(e)) = e$ and $vu = f$, so (i) $\Rightarrow$ (v).

$\varphi(eJ) = veJ \subseteq fR \cap J = fJ$ and $\varphi^{-1}(fJ) = ufJ \subseteq eJ$, so $\varphi(eJ) = fJ$. Clearly now φ induces an isomorphism $\frac{eR}{eJ} \to \frac{fR}{fJ}$, so (i) $\Rightarrow$ (iii). Suppose $u, v \in R$ with $uv = e$ and $vu = f$. Then $ufv = uvuv = e^2 = e$ and similarly $f = veu \in veR$. It is now easy to see that the map $eR \to fR$ given by left multiplication by fv is an R-isomorphism, so (vi) $\Rightarrow$ (i). Suppose $\frac{eR}{eJ} \cong \frac{fR}{fJ}$, so $\frac{eR + J}{J} \cong \frac{fR + J}{J}$. Since (i) $\Rightarrow$ (v) is established there are elements $u \in eRf$, $v \in fRe$ with $uv \equiv e \bmod (J)$ and $vu \equiv f \bmod (J)$. Let $j = uv - e \in J$. $v \in Re$ so $uv = uve = (1 + j)e$, and putting $y = (1 + j)^{-1}$, $e = yuv$. Similarly, $vuz = f$ for some $z \in R$. Hence $e = yufv = yuvuzv = euzv = uzv$, so $v(uz) = f$ and $(uz)v = e$, whence (iii) $\Rightarrow$ (vi). Since (vi) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follows by symmetry and (v) $\Rightarrow$ (vi) is trivial, we have established the equivalence of (i) to (vi). Now suppose e, f are local idempotents. Since $e, f \notin J$, (v) $\Rightarrow$ (vii) and (v) $\Rightarrow$ (viii) are trivial. By 1.2.10, $\frac{eR}{eJ}$ is a simple right R-module, so (vii) $\Rightarrow$ (iii) follows from 1.2.11. (viii) $\Rightarrow$ (iv) follows by symmetry, completing the proof.

1.5.2 <u>COROLLARY</u>:

Suppose $a_1, .. a_n$ are elements of a ring R such that $e = a_1 a_2 .. a_n$ is an idempotent. Then for $1 \leq i \leq n$ $f_i = a_i .. a_n e a_1 .. a_{i-1}$ is an idempotent and $f_i R \cong eR$.

<u>Proof</u>: $f_i^2 = a_i .. a_n e a_1 .. a_n e a_1 .. a_{i-1} = a_i .. a_n e^3 a_1 .. a_{i-1} = f_i$, and clearly $a_1 .. a_{i-1} f_i a_i .. a_n = e^3 = e \neq 0$, so $f_i \neq 0$. The result now follows from 1.5.1 (vi) $\Rightarrow$ (i).

1.5.3 COROLLARY:

Let R be a ring and let $a \in R$ be an idempotent modulo J. Then a can be lifted over J if and only if there is an idempotent $e \in R$ such that $\frac{aR + J}{J} \cong \frac{eR + J}{J}$.

Proof: If a can be lifted over J the result is trivial, so suppose e is an idempotent and $\frac{aR + J}{J} \cong \frac{eR + J}{J}$. By 1.5.1 there are elements $u \in aRe$, $v \in eRa$ with $uv \equiv a \bmod (J)$ and $vu \equiv e \bmod (J)$. Let $j = vu - e \in J$. $u \in Re$ so $vu = vue = (1 + j)e$ and $e = (1 + j)^{-1}vu$. Let $f = u(1 + j)^{-1}v$, an idempotent by 1.5.2. Now $u \equiv u(1 + j) \bmod (J)$, so $u(1 + j)^{-1} \equiv u \bmod (J)$, whence $f \equiv uv \equiv a \bmod (J)$ as required.

1.5.4 THEOREM:

A ring R is semi-perfect if and only if $R = \sum_{i \in I} e_i R$ for some collection $\{e_i\}_{i \in I}$ of local idempotents.

Proof: If R is semi-perfect the result follows from 1.4.4(iii), so suppose $\{e_i\}_{i \in I}$ is a collection of local idempotents and $R = \sum_{i \in I} e_i R$. Therefore $\frac{R}{J} = \sum_{i \in I} \frac{e_i R + J}{J}$. But for each $i \in I$, $\frac{e_i R + J}{J} \cong \frac{e_i R}{e_i J}$, a simple right R-module, hence a simple right $\frac{R}{J}$ - module, so by 1.2.2 $\frac{R}{J}$ is Artinian. Now suppose $a \in R$ is a local idempotent modulo J. $aR = aR(\sum_{i \in I} e_i R) \not\subseteq J$, so $aRe_i \not\subseteq J$ for some $i \in I$. But now by 1.2.11, $\frac{aR + J}{J} \cong \frac{e_i R + J}{J}$. It follows from 1.5.3 that local idempotents can be lifted over J. As in the proof of 1.4.3, we can now lift any idempotent over J, so (by 1.4.3) R is semi-perfect.

Suppose $e_1,..,e_{2n}$ are idempotents in a ring R with $e_1R = e_2R$, $Re_2 = Re_3$, $e_3R = e_4R$, ... $e_{2n-1}R = e_{2n}R$. Then by 1.5.1, $e_1R \cong e_2R \cong ... \cong e_{2n}R$. It is natural to ask when, given idempotents e, f $\in$ R with eR $\cong$ fR, we can find idempotents $e_1,..,e_{2n} \in R$ as above with $e = e_1$, $f = e_{2n}$. Before answering this question, we note the following example.

Let F be a field, and let R be the 2x2 matrix ring over F, a simple Artinian ring. Let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$,

so clearly e, f are local idempotents, eR $\cong$ fR, and ef = fe = 0. If g is an idempotent and eR = gR, Rg = Rf, then $g \in RggR = RfeR = 0$, a contradiction. However, we will see that there exist idempotents g, h $\in$ R with eR = gR, Rg = Rh, and hR = fR.

1.5.5 LEMMA:

Let e, f be primitive idempotents in a ring R. Then the following conditions are equivalent:-

(i) e = fe

(ii) f = ef

(iii) $e \in fRe$

(iv) $f \in eRf$

(v) eR = fR

Proof: (v) $\Rightarrow$ (i) $\Leftrightarrow$ (iii), (v) $\Rightarrow$ (ii) $\Leftrightarrow$ (iv) and (i) + (ii) $\Rightarrow$ (v) are trivial, so it suffices to prove (i) $\Rightarrow$ (ii), whence (ii) $\Rightarrow$ (i) will follow by symmetry, completing the proof. Suppose then e = fe. Then $[f(1 - e)f]^2 = f(1 - e)f \in fRf$. But f is primitive, so by 1.2.10 either $f = f(1 - e)f$ or $f(1 - e)f = 0$. If $f = f(1 - e)f$ then fef = 0, and since e = fe, ef = 0. Therefore $e = (fe)^2 = 0$, a contradiction. Hence $0 = f(1 - e)f = f - fef = f - ef$; i.e. $f = ef$ as required.

1.5.6 PROPOSITION:

Let e, f be local idempotents in a ring R. Then the following conditions are equivalent:-

(i) $fe \notin J$

(ii) eRf contains an idempotent.

(iii) There is an idempotent $g \in R$ with $eR = gR$, $Rg = Rf$.

Proof: Suppose $fe \notin J$, so $feR \nsubseteq fJ$ and by 1.2.10 $feR = fR$. Therefore $f = fex$ for some $x \in R$, and by 1.5.2 exf is an idempotent. So $exfR \nsubseteq eJ$, $Rexf \nsubseteq Jf$, and by 1.2.10 $exfR = eR$, $Rexf = Rf$. Hence (i) $\Rightarrow$ (iii). (iii) $\Rightarrow$ (ii) is trivial. Finally, suppose $g \in eRf$ is an idempotent. Then $g \notin J$ and $g = g^2 \in eRfeRf$, so $fe \notin J$. Hence (ii) $\Rightarrow$ (i) as required.

1.5.7 THEOREM:

Let e, f be local idempotents of a ring R. Then $eR \cong fR$ if and only if there are idempotents $g, h \in R$ such that $eR = gR$, $Rg = Rh$, and $hR = fR$.

Proof: Suppose $eR \cong fR$. If $fe \notin J$ the result follows from 1.5.6, so suppose $fe \in J$. By 1.5.1 $fRe \nsubseteq J$, so $fxe \notin J$ for some $x \in R$. Let $h = f + fxe(1 - f)$. Clearly $h = h^2$, $fh = h$, and $hf = f \neq 0$ (so $h \neq 0$). Therefore, $fR = hR$. Now $fe \in J$, $fxe \notin J$, so $he = fe + fxe(1 - fe) \notin J$. The result now follows from 1.5.6. The converse is an immediate consequence of 1.5.1.

1.5.8 THEOREM:

Let e be a local idempotent of a ring R, and let I be the set of all idempotents $f \in R$ with $fR \cong eR$. Then
$$ReR = \sum_{f \in I} fR = \sum_{f \in I} Rf.$$

Proof: Let $A = \sum_{f \in I} fR$, and suppose $x \in R$. Clearly $xA \leq A$ if and only if $xf \in A$ for each $f \in I$. Suppose then $f \in I$. Now if $xf \notin J$ let $y = xf$, and if $xf \in J$ let $y = f + xf$. Now $y \in Rf$, $y \notin J$, so by 1.2.10 $Ry = Rf$. Thus $f = zy$ for some $z \in R$. Now by 1.5.2 $yz \in I$, so $y = yf = yzy \in yzR \leq A$, so clearly $xf \in A$. Hence A is an ideal. If $f \in I$, $fR \cong eR$ so by 1.5.1 there are elements $u \in fRe$, $v \in eRf$ with $f = uv \in ReR$. So $A \subseteq ReR$. But $e \in A$, an ideal of R. Thus $A = ReR$. The result now follows by symmetry.

We now turn our attention to the socles of a ring R. Suppose M is a minimal right ideal of R and $x \in R$. xM is an R-homomorphic image of M, so clearly $xM = 0$ or xM is a minimal right ideal of R. Hence E_r is an ideal of R. Now by Nakayama's Lemma (1.1.4), $MJ = 0$, so clearly $E_r \leq l(J)$. But $l(J)$ is a right $\frac{R}{J}$ - module, so if $\frac{R}{J}$ is Artinian then by 1.2.2 $l(J) \leq E_r$, so $l(J) = E_r$. Hence we have

1.5.9 LEMMA:

Let R be a ring. Then E_r is an ideal of R, $E_r \leq l(J)$, and if $\frac{R}{J}$ is Artinian then $E_r = l(J)$.

It is well known that if M is a minimal right ideal of a ring R and $M^2 \neq 0$, then $M = eR$ for some idempotent $e \in R$ (see [17], lemma 1.3.1), and clearly $eJ = 0$, so by 1.2.10

e is a local idempotent. Using this result we can now prove

1.5.10 THEOREM:

Let R be a semi-perfect ring. Then R can be uniquely expressed as a direct sum $S \oplus T$ of a semi-simple Artinian ring S and a ring T in which $E_r(T) \cap E_l(T) \leq W(T)$.

Proof: Suppose e is a local idempotent, $e \in E_r \cap E_l$, and suppose f is an idempotent, $fR \subset ReR$. Now $ReR = fR \oplus (1 - f)ReR$, so $(1 - f)Re \neq 0$. But $e \in E_l = r(J)$, so $Je = 0$. Hence $(1 - f)Re \nsubseteq J$. Thus there is a local idempotent $g \in (1 - f)Re$, so $g = (1 - f)g$. By 1.5.2, $g(1 - f)$ is also an idempotent, and clearly f, $g(1 - f)$ are mutually orthogonal. But $g \in (1 - f)Re$ so $g(1 - f) \in ReR$, so $f + g(1 - f)$ is an idempotent in ReR. Since R is semi-perfect, induction shows $ReR = fR$ for some idempotent $f \in R$. Similarly $ReR = Rg$ for some idempotent $g \in R$, so $fR = Rg$, whence $f = fg = g$. Therefore, $ReR = fR = Rf = fRf$, an (ideal) direct summand of R. Since $fR \subseteq E_r \cap E_l$, fR is completely reducible, so by 1.2.2 fR is a semi-simple Artinian ring. The result now follows easily by induction.

1.5.11 Definition: An idempotent f in a ring R is said to be a generating idempotent of R if $R = RfR$.

1.5.12 Definition: Let R be a semi-perfect ring. Then a representative set of idempotents of R is an indexed set $\{e_{ij} : 1 \leq i \leq n,\ 1 \leq j \leq t_i\}$ of mutually orthogonal primitive idempotents of R, for some $n, t_1, \ldots, t_n \in \mathbb{N}$, such that $1 = \sum_{i=1}^{n} \sum_{j=1}^{t_i} e_{ij}$ and $e_{ij}R \cong e_{kl}R \Leftrightarrow i = k$. Clearly any semi-perfect ring R has such a set, and (by 1.5.8) $e_{11} + e_{21} + \cdots + e_{n1}$ is a generating idempotent of R.

1.5.13 PROPOSITION:

Let f be a generating idempotent of a ring R. Then R is semi-perfect if and only if fRf is semi-perfect.

Proof: Suppose e is an idempotent in R, $e \in fRf$. Clearly $eRe = efRfe$, so by 1.2.10 e is local in R if and only if e is local in fRf. If R or fRf is semi-perfect, then $f = e_1 + \cdots + e_n$ for some mutually orthogonal local idempotents $e_1, \ldots, e_n \in fRf$, so clearly $fRf = e_1fRf + \cdots + e_nfRf$ and $R = RfR = Re_1R + \cdots + Re_nR$. The result now follows immediately from 1.5.8 and 1.5.4.

1.5.14 Definition: Let P be a property for rings. P is said to be a Morita invariant property if whenever R is a ring with property P, f is a generating idempotent of R, and $n \in \mathbb{N}$, then both fRf and the nxn matrix ring R_n have property P.

1.5.15 LEMMA:

Let R be a ring and $n \in \mathbb{N}$. Then there is a generating idempotent f of the nxn matrix ring R_n with $fR_nf \cong R$.

Proof: Put $f = (a_{ij}) \in R_n$ where $a_{11} = 1$ and $a_{ij} = 0$ for $(i,j) \neq (1,1)$.

Clearly now 1.5.13 says that being semi-perfect is a Morita invariant property. This is a well-known result that follows easily from well-established theory of Morita invariance (see, for example, [27]). Since we are able to use direct methods to see if the properties discussed in this thesis are Morita invariant, we omit any detailed discussion of the (deep) theory of Morita invariance. Our next theorem also follows easily from this theory, although the proof quoted also uses direct methods.

1.5.16 THEOREM:

Let f be a generating idempotent of a semi-perfect ring R. Then R is right self-injective if and only if fRf is right self-injective.

Proof: See [4], theorem 1.1 .

We end this section by introducing continuous rings, as defined and studied by Utumi in (for example) [40].

1.5.17 Definition: A ring R is right (respectively left) continuous if

(i) Every non-zero right (respectively left) ideal of R has an idempotently generated essential extension in R.

(ii) If $x \in R$ and xR (respectively Rx) is isomorphic to an idempotently generated right (respectively left) ideal of R, then xR (respectively Rx) is idempotently generated.

R is said to be continuous if R is both right and left continuous.

Let R be a right self-injective ring and suppose e is an idempotent of R, $x \in R$, and $h: xR \to eR$ is an isomorphism. By 1.3.11 there are elements $u, v \in R$ with $h^{-1}(e) = ue$ and $e = h(ue) = vue$. Then $ue = ue^2 = uevue$. Now by 1.5.2 uev is an idempotent, and clearly $uevR \subseteq ueR = xR = uevueR \subseteq uevR$.

Hence xR is generated by the idempotent uev. Combining this with 1.3.12 we have, as observed by Utumi in [40],

1.5.18 LEMMA:

A right self-injective ring is right continuous.

§ 6 Noetherian rings.

1.6.1 Definitions: A ring R is right Noetherian if it has the ascending chain condition on right ideals, i.e. if for any chain $I_1 \subseteq I_2 \subseteq \dots$ of right ideals of R, there exists $n \in \mathbb{N}$ (depending on the chain) with $I_n = I_{n+1} = \dots$. R is left Noetherian if it has the ascending chain condition on left ideals, and is Noetherian if it is both right and left Noetherian.

It is straightforward to verify the equivalence of the following conditions on a ring R:-

(i) R is right Noetherian.

(ii) Every non-empty collection of right ideals of R has a maximal element.

(iii) Every right ideal of R is finitely generated.

It is well known that if R is a right or left Noetherian ring then W is nilpotent, so $\frac{R}{W}$ is a semi-prime ring. This fact follows from:-

1.6.2 THEOREM (Levitzki):

Every nil one-sided ideal of a right Noetherian ring is nilpotent.

Proof: See [11], page 49.

1.6.3 THEOREM:

A ring R is right Artinian if and only if R is right Noetherian and $\frac{R}{W}$ is Artinian.

Proof: See [11].

1.6.4 Definitions: A proper ideal P of a ring R is said to be a prime ideal of R if whenever A, B are ideals of R and $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. Clearly every maximal ideal of R is a prime ideal of R. A minimal prime ideal of R is a prime ideal which does not strictly contain any other prime ideal of R. R is said to be a prime ring if 0 is a prime ideal of R.

If A is an ideal of a ring R, it is easy to see that the only prime ideals of $\frac{R}{A}$ are precisely those ideals of the form $\frac{P}{A}$ for some prime ideal P of R containing A.

1.6.5 PROPOSITION:

Let R be a right Noetherian ring. Then R has a finite number of minimal prime ideals, $P_1, .., P_n$ say, and $W = P_1 \cap \ldots \cap P_n$, and no P_i is redundant in this expression.

Proof: See [13], 2.17 .

If R is a right Noetherian ring then W is nilpotent, so it is clear from 1.6.5 that there is a product of prime ideals which equals 0. Applying this to R and the factor rings of R, we immediately get the following well-known result.

1.6.6 LEMMA:

If R is a right Noetherian ring then

(i) Every ideal of R contains a product of prime ideals.

(ii) Every non-zero ideal contains a product of non-zero prime ideals.

1.6.7 Definitions: A proper ideal T of a ring R is said to be a right primary ideal of R if whenever A, B are ideals of R and $AB \subseteq T$, then $A \subseteq T$ or $B^n \subseteq T$ for some $n \in \mathbb{N}$. T is a left primary ideal if whenever A, B are ideals of R and $AB \subseteq T$, then $A^m \subseteq T$ for some $m \in \mathbb{N}$ or $B \subseteq T$. T is a primary ideal if it is both left and right primary. Clearly a prime ideal of R is also primary. R is said to be a primary ring if 0 is a primary ideal of R.

1.6.8 Remark: Let R be a primary Noetherian ring and suppose A, B are ideals of R and $AB \subseteq W$. Then for some $n \in \mathbb{N}$, $(AB)^n = 0$, $(AB)^{n-1} \neq 0$. If $A \not\subseteq W$ then 0 is primary and $AB(AB)^{n-1} = 0$, so $B(AB)^{n-1} = 0$, whence $B \subseteq W$. Hence W is a prime ideal of R. It follows that our definition of a primary ring (1.6.7) is consistent with our definition of a primary Artinian ring (1.2.6).

1.6.9 Definitions: A proper right (respectively left) ideal I of a ring R is said to be a meet-irreducible right (respectively left) ideal of R if whenever X_1, X_2 are right (respectively left) ideals of R and $I = X_1 \cap X_2$, then $X_1 = I$ or $X_2 = I$. Similarly, a meet-irreducible ideal of R is a proper ideal of R that cannot be expressed as an intersection of two strictly larger ideals of R.

1.6.10 LEMMA:

Let R be a right Noetherian ring. Then

(i) Every proper right ideal of R can be expressed as a finite intersection of meet-irreducible right ideals of R.

(ii) Every proper ideal of R can be expressed as a finite intersection of meet-irreducible ideals of R.

<u>Proof</u>: If X_1, X_2 are right ideals with the property described in (i), then clearly $X_1 \cap X_2$ also has this property. Since a proper right ideal I without this property cannot be meet-irreducible, it follows that I is contained in a larger proper right ideal without this property. But R is right Noetherian, so clearly (i) holds. (ii) follows similarly.

1.6.11 <u>Definition</u>: A ring R is said to have <u>primary decomposition</u> if every ideal of R is expressible as a finite intersection of primary ideals of R. If R is right Noetherian, then 1.6.10 shows that R has primary decomposition if and only if every meet-irreducible ideal of R is primary.

1.6.12 <u>Definition</u>: Two ideals A, B of a ring R are said to be <u>co-maximal</u> if A + B = R. More generally, a set $\{A_i\}_{i\in I}$ of ideals of R is said to be a set of co-maximal ideals if $A_i + A_j = R$ for each $i, j \in I$, $i \neq j$.

1.6.13 <u>LEMMA</u>:

Let $A_1,..,A_n$ be co-maximal ideals of a ring R, and suppose $k_1,..,k_n \in \mathbb{N}$. For i = 1,..,n let

$B_i = A_1^{k_1} \cap \ldots \cap A_{i-1}^{k_{i-1}} \cap A_{i+1}^{k_{i+1}} \cap \ldots \cap A_n^{k_n}$. Then

$B_1 + .. + B_n = R = B_i + A_i^{k_i}$ for $1 \leq i \leq n$. Further, if

$A_1^{k_1} \cap \ldots \cap A_n^{k_n} = 0$ then $B_1 \oplus .. \oplus B_n = R = B_i \oplus A_i^{k_i}$ for $1 \leq i \leq n$.

Proof: Suppose M is a maximal, hence prime, ideal of R and $B_1 + .. + B_n \subseteq M$. Then $B_1 \subseteq M$ and M is prime so $A_i \subseteq M$ for some $i \neq 1$. But $B_i \subseteq M$, so $A_j \subseteq M$ for some $j \neq i$. Thus $R = A_i + A_j \subseteq M$, a contradiction. So $B_1 + .. + B_n = R$. If $1 \leqslant i \leqslant n$ then clearly $B_1 + .. + B_{i-1} + B_{i+1} + .. + B_n \subseteq A_i^{k_i}$, whence $B_i + A_i^{k_i} = R$. The final statement of the lemma follows trivially.

Let R be an Artinian ring. By 1.6.8 every primary ideal of R contains a power of a maximal ideal of R, so if R has primary decomposition then there are maximal ideals $M_1,..,M_n$ of R and $k_1,..,k_n \in \mathbb{N}$ such that $M_1^{k_1} \cap \ldots \cap M_n^{k_n} = 0$. Thus the following well-known result follows easily from 1.6.13 .

1.6.14 PROPOSITION:

Let R be an Artinian ring. Then R has primary decomposition if and only if R can be expressed as a direct sum of primary Artinian rings.

1.6.15 Definitions: An element c of a ring R is said to be right regular if $r(c) = 0$, and left regular if $l(c) = 0$. If $r(c) = l(c) = 0$, we say c is regular.

1.6.16 Notation: Let A be a proper ideal of a ring R. Then

$$C_R(A) = \left\{c \in R: c + A \text{ is regular in } \frac{R}{A}\right\}.$$

When it is clear which ring R is under consideration, we write $C(A)$ instead of $C_R(A)$. In particular, $C(0)$ is the set of all regular elements of R.

1.6.17 Definitions: A ring Q is said to be a quotient ring if every regular element of Q is a unit in Q.

Let R be a subring of a ring Q. Then Q is said to be a right (respectively left) quotient ring of R if the regular elements of R are units in Q, and every element of Q can be written in the form ac^{-1} (respectively $c^{-1}a$) for some $a \in R$, $c \in C_R(0)$. In this case it is clear that Q is a quotient ring, and if R is a quotient ring then R = Q. If Q is both a right and a left quotient ring of R, then we say Q is a quotient ring of R. It is easy to verify that if Q_1, Q_2 are both quotient rings of a ring R, then $Q_1 \cong Q_2$, and if, further, Q is a left quotient ring of R, then so are Q_1 and Q_2, and $Q \cong Q_1$. Thus if R is a ring which has a (right) quotient ring Q (i.e. if Q is a (right) quotient ring of R), then we are justified in referring to Q as the (right) quotient ring of R.

1.6.18 Definition: Let C be a multiplicatively closed set of regular elements of a ring R. Then R is said to satisfy the right (respectively left) Ore condition with respect to C if for each $a \in R$, $c \in C$ there are elements $a_1 \in R$, $c_1 \in C$ such that $ac_1 = ca_1$ (respectively $c_1a = a_1c$).

Suppose Q is the right quotient ring of a ring R, and $a \in R$, $c \in C_R(0)$. Then $c^{-1}a \in Q$, so $c^{-1}a = a_1c_1^{-1}$ for some $a_1 \in R$, $c_1 \in C_R(0)$, whence $ac_1 = ca_1$. Thus R satisfies the right Ore condition with respect to $C_R(0)$. Ore has proved a converse to this fact, that is:-

1.6.19 THEOREM (Ore):

A ring R has a right quotient ring if and only if R satisfies the right Ore condition with respect to C(0).

Proof: See [17], theorem 7.1.1 .

1.6.20 THEOREM (Goldie):

A semi-prime (respectively prime) right Noetherian ring R has a semi-simple (respectively simple) Artinian right quotient ring.

Proof: See [11], theorem 5.4 or [13], theorem 1.37 .

One of the important facts established by Goldie in his proof of 1.6.20 is the following:-

1.6.21 PROPOSITION:

Let R be a semi-prime right Noetherian ring. Then

(i) Every right regular element of R is regular.

(ii) A right ideal I of R is essential if and only if I contains a regular element.

Proof: See [11], theorem 4.8 or [17], lemmas 7.2.3 and 7.2.5 .

1.6.22 THEOREM (Small):

A Noetherian ring R has an Artinian quotient ring if and only if C(0) = C(W).

Proof: See [37], part II, section 1 or [13], theorem 2.7 .

There is a natural generalization of the quotient ring of a ring R which we now introduce (see [13]). This generalization is particularly useful in studying Noetherian rings.

1.6.23 <u>Definition</u>: Let R be a subring of a ring Q, and suppose C is a multiplicatively closed set of regular elements of R. Then Q is said to be a <u>partial right</u> (respectively <u>left</u>) <u>quotient ring of R</u> with respect to C if the elements of C are units in Q, and every element of Q can be written in the form ac^{-1} (respectively $c^{-1}a$) for some $a \in R$, $c \in C$.

Ore's proof of 1.6.19 (see [17]) is easily adapted to this case, giving:-

1.6.24 <u>THEOREM</u>:

Let C be a multiplicatively closed set of regular elements of a ring R. Then R has a partial right quotient ring with respect to C if and only if R satisfies the right Ore condition with respect to C.

We will find the following well-known properties useful when considering (partial) quotient rings.

1.6.25 <u>PROPOSITION</u>:

Let R be a ring and let Q be a partial right quotient ring of R with respect to a multiplicatively closed set C of regular elements of R. Then

(i) If I is a right ideal of R, then

$$IQ = \{xc^{-1} \in Q: x \in I,\ c \in C\}.$$

(ii) If I is a right ideal of Q, then $I = (I \cap R)Q$.

(iii) If A is an ideal of R, then $C \subseteq C_R(A)$ if and only if AQ is an ideal of Q and $A = AQ \cap R$.

Proof: The proofs are easily adapted from the case when $C = C_R(0)$, i.e. when Q is the right quotient ring of R. In this case, (i) and (ii) are proved in [13], 1.36, and (iii) is proved in [25], lemma 4.

Let P be a prime ideal of a Noetherian ring R, and suppose $C_R(P) \subseteq C_R(0)$ and Q is a partial right quotient ring of R with respect to $C_R(P)$. It follows easily from 1.6.25(ii) that Q is right Noetherian, and since $P \cap C_R(P) = \emptyset$, 1.6.25(i)+(iii) show that PQ is a proper ideal of Q. It is easy to see that $C_Q(PQ)$ is a set of units of Q, so by 1.6.20 $\frac{Q}{PQ}$ is a simple Artinian ring. If I is a maximal right ideal of Q, $PQ \not\subseteq I$, then $I + PQ = Q$ so clearly $I \cap C_Q(PQ) \neq 0$. Since $C_Q(PQ)$ is a set of units of Q, $I = Q$, a contradiction. Hence Q is a right Noetherian local ring, and $J(Q) = PQ$. We therefore make the following definition:-

1.6.26 Definition: Let P be a prime ideal of a Noetherian ring R, and suppose $C_R(P) \subseteq C_R(0)$ and Q is a partial right (respectively left) quotient ring of R with respect to $C_R(P)$. Then we say Q is a right (respectively left) localisation of R at P. If Q is both a right and a left localisation of R at P, then we say Q is a localisation of R at P. It is easy to verify that if Q_1, Q_2 are both right localisations of R at P, then $Q_1 \cong Q_2$, and if, further, Q is a left localisation of R at P, then so are Q_1 and Q_2, and $Q \cong Q_1$. Thus if P is a prime ideal of a Noetherian ring R, and if we can (right) localise R at P (i.e. if there is a (right) localisation of R at P), then we are justified in referring to the (right) localisation of R at P. If we can localise R at P, then we denote the localisation of R at P by R_P.

1.6.27 Definitions: Let R be a prime Noetherian ring, and let Q be the quotient ring of R. (Q exists by 1.6.20). An ideal A of R is said to be invertible if there is an R - R sub-bimodule A^{-1} of Q, called the inverse of A, such that $AA^{-1} = A^{-1}A = R$. In this case, if $q \in Q$ and $qA \subseteq R$, then $q \in qR = qAA^{-1} \subseteq RA^{-1} = A^{-1}$, so clearly

$$A^{-1} = \{q \in Q:\ qA \subseteq R\} = \{q \in Q:\ Aq \subseteq R\}.$$

If A is invertible, then A^{-1} always denotes the inverse of A. If every non-zero ideal of R is invertible, then R is said to be a prime Noetherian Asano order.

1.6.28 Definitions: A ring R is right (respectively left) hereditary if every right (respectively left) ideal of R is projective as a right (respectively left) R-module. R is said to be hereditary if it is both left and right hereditary. A hereditary prime Noetherian Asano order is called a Dedekind prime ring.

There are a number of theorems concerning commutative Dedekind prime rings, and many of these have been generalized to non-commutative rings using the following definition. We note that in the commutative case, this definition is trivial.

1.6.29 Definition: A prime ring R is said to be bounded if each essential one-sided ideal of R contains a non-zero (two-sided) ideal of R.

The following is a theorem of Asano ([2], satz 2.12) and Michler ([26], theorem 3.5). A short proof is given by Lenagan in [23].

1.6.30 THEOREM:

A bounded prime Noetherian Asano order is a Dedekind prime ring.

Hajarnavis and Lenagan have also proved the following important result (see [15], theorem 2.6).

1.6.31 THEOREM:

If R is a prime Noetherian ring, then R is an Asano order if and only if we can localise R at each maximal ideal P, and the localisation R_P is hereditary.

1.6.32 Definitions: A ring R is said to be a p.r.i.-ring (principal right ideal ring) if every right ideal of R is principal. Clearly such a ring is right Noetherian. R is a p.l.i.-ring (principal left ideal ring) if every left ideal of R is principal.

In [33], theorem 3.5, Robson proved

1.6.33 THEOREM:

If R is a Dedekind prime ring, then every proper homomorphic image of R is an Artinian p.r.i.- and p.l.i.-ring.

As a partial converse to 1.6.33, Hajarnavis has shown ([14], theorems 3.5 and 2.2):-

1.6.34 THEOREM:

Let R be a bounded prime Noetherian ring such that every proper homomorphic image of R is an Artinian p.r.i.- and p.l.i.-ring. Then R is a Dedekind prime ring.

Theorem 1.6.34 is the starting point of our results in chapter 4, where we study Noetherian rings for which every ideal of every proper factor ring is principal as a right ideal.

Chapter 2.

D-, RD-, AND PD-RINGS.

(Generalizations of Quasi-Frobenius Rings).

In this chapter we will introduce Quasi-Frobenius rings and define three generalizations of these rings, namely D-rings, RD-rings and PD-rings. We will show that for self-injective rings, these three classes of rings all coincide with self-injective self-cogenerator rings, as studied by Osofsky in [30]. Examples relating to topics discussed in this chapter are given in chapter 3, and we will frequently refer to these examples. In particular, chapter 3 contains examples to show that D-rings, RD-rings and PD-rings need not be self-injective, and that these classes of rings do not always coincide. Our definition of a D-ring coincides with the definition of an annihilator ring as given in [36], where Skornjakov proves an interesting result on self-injective D-rings. We generalize this result in § 3, and using Skornjakov's methods, show that finitely generated modules over D-rings are finite dimensional. Several other authors have also proved results applicable to D-rings, particularly in the self-injective case, notably in [19], [21], [22] and [38]. However, we are concerned mainly with arbitrary D-rings, no detailed study of which has previously been undertaken. Our definition of RD-rings and PD-rings is apparently new.

§ 1 Introduction.

We start this section with Nakayama's original definition of a Quasi-Frobenius ring, and two well-known results on such rings.

2.1.1 Definition ([28]): Let R be an Artinian ring with a representative set of idempotents $\{e_{ij}: 1 \leq i \leq n,\ 1 \leq j \leq t_i\}$. R is said to be a Quasi-Frobenius ring if there is a permutation π on $\{1,\ldots,n\}$ such that for each i,

(i) $\dfrac{e_{\pi(i)1}R}{e_{\pi(i)1}J} \cong e_{i1}E_r$, a minimal right ideal of R.

(ii) $\dfrac{Re_{i1}}{Je_{i1}} \cong E_l e_{\pi(i)1}$, a minimal left ideal of R.

2.1.2 THEOREM:

An Artinian ring R is a Quasi-Frobenius ring if and only if

(i) $I = rl(I)$ for every right ideal I of R.

(ii) $L = lr(L)$ for every left ideal L of R.

Proof: See [28], theorem 6.

The following theorem, proved by Eilenberg and Nakayama, [9], and others, demonstrates the connection between Quasi-Frobenius rings and self-injective rings.

2.1.3 THEOREM:

The following conditions on a right Noetherian ring R are equivalent:-

(i) R is a Quasi-Frobenius ring.

(ii) R is right self-injective.

(iii) R is left self-injective.

Proof: See [10], theorem 1.

In [18], Ikeda defined a D_r (respectively D_l) ring to be an Artinian ring in which (i) (respectively (ii)) of theorem 2.1.2 holds. Since then, however, this notation has not been used. We adapt Ikeda's notation, making the natural generalization of Quasi-Frobenius rings suggested by 2.1.2 .

2.1.4 Definition: A D-ring is a ring R in which

(i) $I = rl(I)$ for every right ideal I of R.

(ii) $L = lr(L)$ for every left ideal L of R.

In a D-ring R, we have an 'inverted' duality between the right ideal structure and the left ideal structure of R, given by

2.1.5 Duality (for a D-ring R):

(a) $I \longrightarrow l(I)$ for each right ideal I of R.

(b) $L \longrightarrow r(L)$ for each left ideal L of R.

(This duality is 'inverted' in the sense that if I_1, I_2 are right ideals of R, and $I_1 \subseteq I_2$, then $l(I_1) \supseteq l(I_2)$). This duality, together with the observation that every proper right ideal of R is contained in a maximal right ideal of R, immediately gives

2.1.6 LEMMA:

Let I be a right ideal of a D-ring R. Then

(i) I is a minimal right ideal $\Leftrightarrow$ $l(I)$ is a maximal left ideal of R.

(ii) I is a maximal right ideal $\Leftrightarrow$ $l(I)$ is a minimal left ideal of R.

(iii) Every non-zero right ideal of R contains a minimal right ideal, i.e. E_r is an essential right ideal.

Suppose $\{I_x\}_{x\in X}$ is a collection of right ideals of a ring R. Then clearly $l(\sum_{x\in X} I_x) = \bigcap_{x\in X} l(I_x)$, a fact we will use frequently. We complete our initial observations on D-rings by showing a dual relation to this.

2.1.7 LEMMA:

Let $\{I_x\}_{x\in X}$ be a collection of right ideals of a D-ring R. Then $l(\bigcap_{x\in X} I_x) = \sum_{x\in X} l(I_x)$.

Proof: $l(\bigcap_{x\in X} I_x) = l(\bigcap_{x\in X} rl(I_x)) = lr(\sum_{x\in X} l(I_x)) = \sum_{x\in X} l(I_x)$ as required.

2.1.8 Definition: A ring R is said to be a right RD-ring (right restricted D-ring) if

(i) $I\cap r(J) = rl(I)\cap r(J)$ for each right ideal I of R.

(ii) $L + J = lr(L) + J$ for each left ideal L of R.

Similarly, R is a left RD-ring if

(a) $L\cap l(J) = lr(L)\cap l(J)$ for each left ideal L of R.

(b) $I + J = rl(I) + J$ for each right ideal I of R.

R is said to be an RD-ring if R is both a right and a left RD-ring. Clearly a D-ring is also an RD-ring.

Let R be a right RD-ring, and let I and L be right and left ideals of R respectively. If $J \subseteq L$, then $J \subseteq L \subseteq lr(L)$, so $L = L + J = lr(L) + J = lr(L)$. Similarly, if $I \subseteq r(J)$, then $rl(I) \subseteq rlr(J) = r(J)$, so $I = I \cap r(J) = rl(I) \cap r(J) = rl(I)$. Suppose $I \cap r(J) = 0$, so $r(l(I) + J) = rl(I) \cap r(J) = I \cap r(J) = 0$. Thus $l(I) + J = lr(l(I) + J) = R$. But J is superfluous, so $l(I) = R$, and thus $I = 0$. Hence $r(J)$ is an essential right ideal of R (a dual property to the fact that J is a superfluous left ideal of R).

In 2.2.2 we show that right RD-rings are semi-perfect. In anticipation of this result, we now make the following definition.

2.1.9 <u>Definition</u>: A ring R is said to be a <u>right PD-ring</u> (right partial D-ring) if R is semi-perfect and

(i) $I = rl(I)$ for each right ideal I contained in $r(J)$.

(ii) $L = lr(L)$ for each left ideal L containing J.

(iii) $r(J)$ is an essential right ideal of R.

A <u>left PD-ring</u> is defined in the obvious way, and a right and left PD-ring is called a <u>PD-ring</u>.

If L is a left ideal of a ring R with $J \subseteq L$, then clearly $r(L) \subseteq r(J)$, and dually, if I is a right ideal of R, $I \subseteq r(J)$, then clearly $J \subseteq lr(J) \subseteq l(I)$. It follows that if R is a right RD-ring or a right PD-ring, then we have a duality between the right ideals of R contained in $r(J)$ and the left ideals of R containing J, given by

2.1.10 Duality (for a right RD-ring or a right PD-ring R):-

(a) $I \rightarrow l(I)$ for each right ideal I contained in $r(J)$.

(b) $L \rightarrow r(L)$ for each left ideal L containing J.

2.1.11 LEMMA:

Let R be a right RD-ring or a right PD-ring, and let I and L be a right and a left ideal of R respectively. Then

(i) I is a minimal right ideal of R $\Leftrightarrow$ $l(I)$ is a maximal left ideal of R.

(ii) L is a maximal left ideal of R $\Rightarrow$ $r(L)$ is a minimal right ideal of R.

(iii) $E_r = r(J)$, an essential right ideal of R.

Proof: Since $r(J)$ is an essential right ideal of R, $r(J)$ contains every minimal right ideal of R, i.e. $E_r \subseteq r(J)$. (i) and (ii) now follow from the duality 2.1.10. But E_r is the sum of all minimal right ideals, so by (i) and (ii), $l(E_r)$ is the intersection of all maximal left ideals of R, i.e. $l(E_r) = J$. Therefore, $E_r = rl(E_r) = r(J)$ as required.

In an analogous way to 2.1.7, we can complete our initial observations on RD-rings and PD-rings with the following lemma.

2.1.12 LEMMA:

Let R be a right PD-ring. Then

(i) If $\{I_x\}_{x\in X}$ is a collection of right ideals of R, each contained in $r(J)$, then $l(\bigcap_{x\in X} I_x) = \sum_{x\in X} l(I_x)$.

(ii) If $\{L_x\}_{x\in X}$ is a collection of left ideals of R, each containing J, then $r(\bigcap_{x\in X} L_x) = \sum_{x\in X} r(L_x)$.

Further, if R is a right RD-ring, then (i) and (ii) above hold, in addition to

(iii) If $\{I_x\}_{x\in X}$ is a collection of right ideals of R, then

$$l(\bigcap_{x\in X} I_x) + J = \sum_{x\in X} l(I_x) + J.$$

Proof: (i) and (ii) are straightforward and left to the reader (c.f. 2.1.7). Suppose R is a right RD-ring, and $\{I_x\}_{x\in X}$ is a collection of right ideals of R. Then

$$r(\sum_{x\in X} l(I_x) + J) = \bigcap_{x\in X} rl(I_x)\cap r(J) = \bigcap_{x\in X} I_x \cap r(J)$$
$$= rl(\bigcap_{x\in X} I_x)\cap r(J) = r(l(\bigcap_{x\in X} I_x) + J),$$

and (iii) follows immediately.

We will observe some properties of right RD-rings and right PD-rings in the next section, but in this chapter we are mainly concerned with rings with two-sided conditions. In this section, we have defined four such classes of rings, namely

Class 1: Quasi-Frobenius rings.

Class 2: D-rings.

Class 3: RD-rings.

Class 4: PD-rings.

In 2.2.2 we show that right RD-rings are semi-perfect, and hence are right PD-rings. Clearly now

class 1 $\subseteq$ class 2 $\subseteq$ class 3 $\subseteq$ class 4.

We will see (in 2.2.9) that PD-rings are a natural generalization of Nakayama's original definition of Quasi-Frobenius rings (2.1.1), and will deduce that in the Artinian case (in fact even the right Noetherian case - see 2.2.10) these four classes coincide. In 2.2.18 we show that RD-rings are continuous PD-rings, and conversely. Continuous rings are a generalization of self-injective rings, and in 2.2.14 we show that in the self-injective case, classes 2, 3 and 4 each coincide with the class of self-injective self-cogenerator rings (defined in 2.2.11) as studied by Osofsky in [30]. However, in 3.2.2 we give an example of a D-ring which is not self-injective, showing that class 1 $\neq$ class 2. In 3.2.3 we give an example to show class 2 $\neq$ class 3, and in 3.2.5 we show class 3 $\neq$ class 4.

§ 2 Characterizations of D-, RD-, and PD-rings.

2.2.1 THEOREM:

Let R be a ring such that $l(I \cap r(J)) = l(I) + J$ for each right ideal I of R. Then idempotents can be lifted over J.

Proof: Suppose $a \in R$ and $a - a^2 \in J$. By Zorn's lemma, there is a right ideal I of R maximal with respect to the property $I \cap r(J) \subseteq r(a)$. Now $a \in lr(a) \subseteq l(I \cap r(J)) = l(I) + J$, so there is an element $b \in l(I)$ with $b \equiv a \equiv a^2 \equiv b^2 \bmod (J)$. Now $b^2 \in lr(b^2)$ and $a - b^2 \in J$, so $a \in lr(b^2) + J$, and hence $r(b^2) \cap r(J) = r(lr(b^2) + J) \subseteq r(a)$. But $b \in l(I)$ so $I \subseteq rl(I) \subseteq r(b) \subseteq r(b^2)$, and by the maximality of I, $I = r(b) = r(b^2)$. Suppose $x \in R$ and $bx \in r(b)$, so $b^2x = 0$. Then $x \in r(b^2) = r(b)$, i.e. $bx = 0$. Therefore, $bR \cap r(b) = 0$. Now $b(1 - b) \in J$, so $(1 - b)r(J) \subseteq r(b)$, and hence $(1 - b)(bR \cap r(J)) \subseteq bR \cap r(b) = 0$. Clearly now $bR \cap r(J) = br(J)$. But $a - b \in J$, so $bR \cap r(J) = br(J) = ar(J)$. Now $a - a^2 \in J$, so $(1 - a) - (1 - a)^2 \in J$, and in the same way as the above we can find an element $c \in R$ with $c \equiv 1 - a \bmod (J)$ and $cR \cap r(J) = (1 - a)r(J)$. If $x \in r(J)$, then $a - a^2 \in J$ so $ax = a^2x$. Hence $ar(J) \cap (1 - a)r(J) \subseteq a(1 - a)r(J) \subseteq Jr(J) = 0$, so $bR \cap cR \cap r(J) = 0$. But now

$$\begin{aligned} R &= l((bR \cap cR) \cap r(J)) \\ &= l(bR \cap cR) + J, \end{aligned}$$

and since J is superfluous in R, $R = l(bR \cap cR)$, so $bR \cap cR = 0$. Now $b - a \in J$ and $c - (1 - a) \in J$, so $(bR \oplus cR) + J = R$, and since J is superfluous, $R = bR \oplus cR$. Now by 1.2.8, $bR = eR$ for some $e = e^2 \in R$, and since $b - a \in J$, $aR + J = bR + J = eR + J$. It follows from 1.2.16 that a can be lifted over J, as required.

2.2.2 THEOREM:

A right RD-ring is semi-perfect, and hence is a right PD-ring.

Proof: Let R be a right RD-ring and let I be a right ideal of R. Then $l(I \cap r(J)) = l(rl(I) \cap r(J)) = lr(l(I) + J) = l(I) + J$, so by 2.2.1 idempotents can be lifted over J. Now by 1.1.2, $E_r = \sum_{x \in X} \oplus M_x$ for some collection $\{M_x\}_{x \in X}$ of minimal right ideals of R. For each $y \in X$, let $F_y = \sum_{\substack{x \in X \\ x \neq y}} M_x$. Clearly $\bigcap_{x \in X} F_x = 0$, so by 2.1.12 $R = \sum_{x \in X} l(F_x)$. Now by 2.1.11, $E_r = r(J)$, so $J = l(E_r)$. But for each $x \in X$, $\frac{E_r}{F_x} \cong M_x$, a simple right R-module, so by the duality 2.1.10 $\frac{l(F_x)}{l(E_r)} = \frac{l(F_x)}{J}$ is a simple left R-module, hence a simple left $\frac{R}{J}$ - module. But $R = \sum_{x \in X} l(F_x)$, so $\frac{R}{J} = \sum_{x \in X} \frac{l(F_x)}{J}$, a completely reducible left $\frac{R}{J}$ - module. Hence by 1.2.2 $\frac{R}{J}$ is semi-simple Artinian, so by 1.4.3, R is semi-perfect. It follows from our remarks following the definition of a right RD-ring (2.1.8) that R is a right PD-ring.

2.2.3 Remarks: In defining right RD-rings and right PD-rings we introduced certain conditions on a ring R. In 2.2.2 we observed that the condition described in 2.2.1 is also satisfied by right RD-rings, so for the purposes of this discussion we number this condition 2.1.8(iii). Hence we have:-

2.1.8(i) $I \cap r(J) = rl(I) \cap r(J)$ for each right ideal I of R.

2.1.8(ii) $L + J = lr(L) + J$ for each left ideal L of R.

2.1.8(iii) $l(I \cap r(J)) = l(I) + J$ for each right ideal I of R.

2.1.9(i) $I = rl(I)$ for each right ideal I contained in $r(J)$.

2.1.9(ii) $L = lr(L)$ for each left ideal L containing J.

2.1.9(iii) $r(J)$ is an essential right ideal of R.

Notice that 2.1.8(i) is a natural dual to 2.1.8(ii), and 2.1.9(i) is a natural dual to 2.1.9(ii). Further, 2.1.8(iii) is a natural dual to the fact that $r(L + J) = r(L) \cap r(J)$ for each left ideal L of R, and 2.1.9(iii) is a natural dual to the fact that J is a superfluous left ideal of R. We have already observed that 2.1.8(i) $\Rightarrow$ 2.1.9(i), 2.1.8(ii) $\Rightarrow$ 2.1.9(ii) and 2.1.8(i) + (ii) $\Rightarrow$ 2.1.8(iii), and it is easy to see 2.1.8(iii) $\Rightarrow$ 2.1.9(iii) (c.f. the proof of 2.2.1). If L is a left ideal of R then $L + J \subseteq lr(L) + J \subseteq lr(L + J)$, so clearly 2.1.9(ii) $\Rightarrow$ 2.1.8(ii). Suppose R satisfies 2.1.8(iii) and 2.1.9(i), and let I be a right ideal of R. Then $I \cap r(J) = rl(I \cap r(J)) = r(l(I) + J) = rl(I) \cap r(J)$, so 2.1.8(iii) + 2.1.9(i) $\Rightarrow$ 2.1.8(i). Hence 2.1.9(i) + (ii) + 2.1.8(iii) $\Leftrightarrow$ 2.1.8(i) + (ii), and it follows that R is a RD-ring if and only if R is a right PD-ring satisfying 2.1.8(iii). Now 2.1.8(iii) is the condition we used to lift idempotents, and in 3.1.7 we give an example of a ring satisfying 2.1.9(i) + (ii) + (iii) in which idempotents cannot be lifted, so this example is not a right PD-ring. Notice that $\mathbb{Z}$, the ring of rational integers, is semi-simple, so trivially satisfies 2.1.8(iii), but $\mathbb{Z}$ is not semi-perfect. In 3.1.3 and 3.1.4 we give examples of two rings satisfying 2.1.8(iii), the first also satisfying 2.1.8(i) and the second satisfying 2.1.8(ii), but neither being semi-perfect.

Hence we need the full force of right RD-rings to prove theorem 2.2.2 . It is this theorem that justifies the terminology 'partial D-ring' used in definition 2.1.9 .

In an analogous way to the Artinian case (see [18]), we now have

2.2.4 PROPOSITION:

Let R be a right PD-ring. Then

(i) $E_r = E_l$ = the sum of all minimal ideals of R.

(ii) If e is a primitive idempotent of R, then eR is a uniform right ideal of R and eE_r is a minimal right ideal of R.

Proof: By 2.1.11 $E_r = r(J)$ and by 1.5.9 $r(J) = E_l$, so $E_r = E_l$. Now E_r is an essential right ideal of R, so clearly every minimal ideal of R is contained in E_r. The duality 2.1.10 now shows that the set of all minimal ideals of R is precisely the set of all right annihilators of maximal ideals of R. But $\frac{R}{J}$ is semi-simple Artinian, so J is the intersection of all maximal ideals of R, so by 2.1.12 $r(J) = E_r$ is the sum of all minimal ideals of R, so (i) holds. Finally, if e is a primitive idempotent of R, then R(1 - e) + J is a maximal left ideal, so $eE_r = eR \cap r(J) = r(R(1 - e) + J)$ is a minimal right ideal of R. Since E_r is an essential right ideal of R, (ii) follows easily.

It follows immediately from 2.2.4 and 1.5.10 that a semi-prime right PD-ring is a semi-simple Artinian ring.

In 3.1.6 we give an example to show that the converse of proposition 2.2.4 does not hold. However, we will see that under symmetric conditions we can establish a converse, enabling us to show that a PD-ring is a natural generalization of a Quasi-Frobenius ring. In preparation, we make the following definition which, according to Kato, is due to F. Kasch.

2.2.5 Definition: A ring R is said to be a right S-ring if every proper left ideal of R has non-zero right annihilator. R is a left S-ring if every proper right ideal of R has non-zero left annihilator, and is an S-ring if it is both a right and a left S-ring.

Since every proper left ideal of a ring R is contained in a maximal left ideal of R, clearly R is a right S-ring if and only if $r(L) \neq 0$ for each maximal left ideal L of R. Hence a right PD-ring is a right S-ring.

We now give two well-known results concerning S-rings.

2.2.6 LEMMA:

Let R be a right S-ring and suppose $\frac{R}{J}$ is semi-simple Artinian. Then $lr(L) + J = L + J$ for each left ideal L of R.

Proof: If L is a maximal left ideal of R, then $L \subseteq lr(L) \neq R$, since $r(L) \neq 0$, so clearly $L = lr(L)$. Suppose now L is any proper left ideal. $\frac{R}{J}$ is semi-simple Artinian, so $L + J = L_1 \cap \ldots \cap L_n$ for some maximal left ideals $L_1, \ldots, L_n$. Now for each i, $L \subseteq L_i$, so $lr(L) \subseteq L_i$. Hence $L + J \subseteq lr(L) + J \subseteq L_1 \cap \ldots \cap L_n = L + J$, so $L + J = lr(L) + J$ as required.

2.2.7 LEMMA:

A ring R is a right S-ring if and only if every simple left R-module is isomorphic to a minimal left ideal of R.

Proof: Suppose R is a right S-ring and M is a simple left R-module. Choose $0 \neq m \in M$, so $M = Rm$. Let $L = \{x \in R: xm = 0\}$, a proper left ideal of R. Now $r(L) \neq 0$, so choose $0 \neq z \in r(L)$ whence clearly $xm = 0 \Rightarrow xz = 0$. Hence we can define an R-epimorphism $f: M \to Rz$ by $f(rm) = rz$ for each $r \in R$. Since M is simple, f is clearly an isomorphism, so $M \cong Rz$, and Rz must be a minimal left ideal of R. Conversely, suppose every simple left R-module is isomorphic to a minimal left ideal of R, and suppose L is a maximal left ideal of R. $\frac{R}{L}$ is a simple left R-module, so there is a R-monomorphism $f: \frac{R}{L} \to R$. Suppose $f(1 + L) = z \neq 0$. For each $l \in L$, $lz = lf(1 + L) = f(l + L) = 0$, so $0 \neq z \in r(L)$. Hence R is a right S-ring as required.

In a similar way to the Artinian case (see [18] and [28]), we now establish two characterizations of PD-rings.

2.2.8 PROPOSITION:

A semi-perfect ring R is a PD-ring if and only if

(i) $E_r = E_l$, and is essential both as a right and as a left ideal of R.

(ii) If e is a primitive idempotent of R, then eR and Re are uniform right and left ideals respectively.

Proof: If R is a PD-ring, conditions (i) and (ii) hold by 2.1.11 and 2.2.4 . Conversely, suppose R is a semi-perfect ring satisfying these conditions. If L is a maximal left ideal of R, $L = R(1 - e) + J$ for some idempotent $e \in R$, so by 1.5.9 $r(L) = eR \cap r(J) = eR \cap E_1 \neq 0$ by (i). Hence R is a right S-ring, so by 2.2.6 $L = lr(L)$ for each left ideal L of R containing J. Suppose M is a minimal right ideal of R. Clearly $Me \neq 0$ for some primitive idempotent $e \in R$, so $me \neq 0$ for some $m \in M$, whence $M = meR$. Now $0 \neq Rme \leq E_r e = E_1 e$, a minimal left ideal of R since Re is uniform. But $Rme \cong \frac{R}{l(me)} = \frac{R}{l(M)}$, so $l(M)$ is a maximal left ideal of R, so $l(M) = R(1 - f) + J$ for some primitive idempotent $f \in R$. Now $M \leq rl(M) = fR \cap r(J) = fE_r$, and since fR is uniform, fE_r is a minimal right ideal of R. Therefore $M = rl(M) = fE_r$. Suppose I is a right ideal of R and $I \leq r(J) = E_1 = E_r$. By 1.1.2, 1.3.3 and the above, there are primitive idempotents $e_1, \ldots, e_n \in R$ with $I = e_1E_r \oplus \ldots \oplus e_nE_r$. Clearly $I = (e_1R \oplus \ldots \oplus e_nR) \cap E_r$ (by 1.3.3), and there is an idempotent $f \in R$ with $(e_1R \oplus \ldots \oplus e_nR) + J = fR + J$. By 1.2.15 we may assume $f \in e_1R + \ldots + e_nR$, so $fE_r = fR \cap E_r \leq I$. A dimension argument clearly gives $fE_r = I$. Hence $l(I) = R(1 - f) + J$, so $rl(I) = fR \cap r(J) = fE_r = I$. Thus R is a right PD-ring, and by symmetry, R is a PD-ring.

2.2.9 THEOREM:

Let R be a semi-perfect ring with a representative set of idempotents $\{e_{ij}: 1 \leq i \leq n,\ 1 \leq j \leq t_i\}$. Then R is a PD-ring if and only if there is a permutation π on $\{1,..,n\}$ such that for each i, $1 \leq i \leq n$,

(i) $e_{i1}R$ is a uniform right ideal of R, and $\frac{e_{i1}R}{e_{i1}J} \cong e_{\pi(i)1}E_r$, a minimal right ideal of R.

(ii) $Re_{\pi(i)1}$ is a uniform left ideal of R, and

$\frac{Re_{\pi(i)1}}{Je_{\pi(i)1}} \cong E_l e_{i1}$, a minimal left ideal of R.

<u>Proof</u>: Suppose R is a PD-ring and $1 \leq i \leq n$. Now $0 \neq e_{i1}E_r$, a minimal right ideal of R by 2.2.4 . By 1.2.11 for each e_{jk},
$e_{i1}E_r e_{jk} \neq 0 \Leftrightarrow e_{i1}E_r \cong \frac{e_{jk}R}{e_{jk}J} \cong \frac{e_{j1}R}{e_{j1}J} \Leftrightarrow e_{i1}E_r e_{j1} \neq 0$.
Hence (by 1.5.1 and 1.5.12) there is a unique $j \in \{1,..,n\}$ with $e_{i1}E_r e_{j1} \neq 0$. Similarly, since $E_r = E_l$ by 2.2.4, for each j there is a unique i ($1 \leq i,j \leq n$) with $e_{i1}E_r e_{j1} \neq 0$. Hence we have a permutation π on $\{1,..,n\}$ defined by $\pi(i) = j$ whenever $e_{i1}E_r e_{j1} \neq 0$. Clearly now (i) holds, and since $E_r = E_l$ 1.2.11 shows that (ii) holds. Conversely, suppose R, $\{e_{ij}\}$ and π are as in the statement of the theorem. If e is a primitive idempotent of R, $eR \not\subseteq J$, so $eRe_{ij} \not\subseteq J$ for some e_{ij}, whence by 1.5.1 $eR \cong e_{ij}R \cong e_{i1}R$ and $Re_{i1} \cong Re$. Hence by (ii), eR and Re are uniform right and left ideals respectively. By 1.2.11 and (ii), $e_{ij}E_l e_{\pi(i)1} \neq 0$ for each e_{ij}, so $e_{ij}E_l \neq 0$. But $e_{ij}R \cong e_{i1}R$, a uniform right ideal containing a (unique) minimal right ideal, so $e_{ij}E_r \subseteq e_{ij}E_l$. Thus $E_r \subseteq E_l$, and by symmetry, $E_r = E_l$. Now $E_r = \sum_{i=1}^{n} \sum_{j=1}^{t_i} \oplus\, e_{ij}E_r$, so

$\dim((E_r)_R) = t_1 + .. + t_n$. But each $e_{ij}R$ is uniform and

$R = \sum_{i=1}^{n} \sum_{j=1}^{t_i} \oplus\, e_{ij}R$, so $\dim(R_R) = t_1 + .. + t_n$. Hence by 1.3.3

E_r is an essential right ideal of R. By symmetry, $E_l = E_r$ is an essential left ideal of R, so by 2.2.8 R is a PD-ring as required.

Theorem 2.2.9 shows that PD-rings can be considered as a natural generalization of Nakayama's original definition of Quasi-Frobenius rings. It is now clear that the classes of Quasi-Frobenius rings, Artinian D-rings, Artinian RD-rings and Artinian PD-rings coincide. In fact we can prove a stronger result than this.

2.2.10 THEOREM:

The following conditions on a right Noetherian ring R are equivalent:-

(i) R is a Quasi-Frobenius ring.

(ii) R is a D-ring.

(iii) R is an RD-ring.

(iv) R is a PD-ring.

Proof: (i) $\Rightarrow$ (ii) is 2.1.2, (ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (iv) is 2.2.2 . Suppose R is a PD-ring and $x \in J$. R is right Noetherian, so $r(x^n) = r(x^{n+1})$ for some $n \in \mathbb{N}$, whence clearly $x^nR \cap r(x) = 0$. But $x \in J$, so $E_r = r(J) \subseteq r(x)$ and E_r is an essential right ideal of R, so $x^nR = 0$, i.e. $x^n = 0$. Hence J is a nil ideal, so by 1.6.2 J is nilpotent. Thus $J = W$. Now R is semi-perfect so $\frac{R}{J}$ is Artinian, whence by 1.6.3 R is right Artinian. Now by 1.2.4 R_R has a composition series, $R = I_0 \supset I_1 \supset \dots \supset I_k = 0$ say. Suppose $0 \leq i < k$. If $x \in l(I_{i+1})$, then $I_{i+1} \subseteq r(x) \cap I_i$. But $xI_i \cong \frac{I_i}{I_i \cap r(x)}$ and $\frac{I_i}{I_{i+1}}$ is a simple right R-module, so $xI_i \subseteq E_r$. Thus $l(I_{i+1})I_i \subseteq E_r$. Now $I_ie \not\subseteq I_{i+1}$ for some primitive idempotent $e \in R$, so $ye \notin I_{i+1}$ for some $y \in I_i$, whence $I_i = yeR + I_{i+1}$.

Thus $l(I_i) = l(ye) \cap l(I_{i+1})$. But now

$$\frac{l(I_{i+1})}{l(I_i)} = \frac{l(I_{i+1})}{l(ye) \cap l(I_{i+1})} \cong l(I_{i+1})ye \subseteq l(I_{i+1})I_i e \subseteq E_r e,$$

a simple left R-module. Hence, eliminating some terms if necessary, the chain $0 = l(I_o) \leq l(I_1) \leq \dots \leq l(I_k) = R$ provides a composition series for ${}_R R$, so by 1.2.4 R is left Artinian. (iv) $\Rightarrow$ (i) now follows from 2.2.9 .

2.2.11 Definition: A ring R is said to be a right (respectively left) self-cogenerator ring if for any right (respectively left) R-module M there is an R-monomorphism from M into a direct product of copies of R. R is said to be a self-cogenerator ring if it is both a left and a right self-cogenerator ring.

Self-injective self-cogenerator rings were studied by Osofsky in [30]. We will show that the class of self-injective self-cogenerator rings coincide with the classes of self-injective D-rings, self-injective RD-rings, and self-injective PD-rings.

2.2.12 LEMMA ([30] lemma 1 and [22] theorem 1):

The following conditions on a right self-injective ring R are equivalent:-

(i) R is a left S-ring.

(ii) R is a right self-cogenerator ring.

Proof: (i) $\Rightarrow$ (ii): Let M be a right R-module and let H be the set of all R-homomorphisms from M_R to R_R. If $0 \neq m \in M$ then by Zorn's Lemma, mR has a maximal submodule K. Now by 2.2.7 there is an R-homomorphism $h':mR \rightarrow R$ with kernel K.

R is right self-injective, so clearly h' extends to an R-homomorphism $h:M \to R$. Thus $h(m) = h'(m) \neq 0$. Clearly now the map $f:M \to \prod_{h\in H} R$ defined by $f(m) = (h(m))_{h\in H}$ is an R-monomorphism as required.

(ii) $\Rightarrow$ (i): Let I be a proper right ideal of R, and suppose $f:\frac{R}{I} \to \prod_{x\in X} R$ is a monomorphism. $f \neq 0$, so for some canonical epimorphism $h:\prod_{x\in X} R \to R$, $0 \neq hf:\frac{R}{I} \to R$.

$0 \neq hf(1 + I) = z$ say. Now for each $i \in I$, $0 = hf(i + I)$

$$= hf(1 + I)i = zi,$$

and so $0 \neq z \in l(I)$. Thus R is a left S-ring, as required.

The next result was first proved by Kato in [22], theorem 1, although the proof given below is due to Björk ([4], proposition 2.1).

2.2.13 <u>THEOREM</u>:

Let R be a right self-injective ring. Then R is a left S-ring if and only if $I = rl(I)$ for each right ideal I of R.

<u>Proof</u>: Suppose R is a left S-ring, I is a right ideal of R and $I \subset rl(I)$. Choose $x \in rl(I)$, $x \notin I$. By Zorn's Lemma, there is a right ideal K of R maximal with respect to $I \leq K \subset xR + I \leq rl(I)$. Now $\frac{xR + I}{K}$ is simple, so by 2.2.7 there is an R-homomorphism $f:xR + I \to R$ with kernel K, so $f(I) = 0$, $f(x) \neq 0$. Now by 1.3.11 there is an element $z \in R$ with $f(y) = zy$ for all $y \in xR + I$. Thus $zI = 0$, i.e. $z \in l(I) = lrl(I) \leq l(xR + I)$. Thus $f = 0$, a contradiction, and $I = rl(I)$ after all. Since the converse is trivial, the result follows.

Since a PD-ring is an S-ring, combining 2.2.12 and 2.2.13 immediately gives

2.2.14 COROLLARY:

The following conditions on a self-injective ring R are equivalent:-

(i) R is an S-ring.

(ii) R is a D-ring.

(iii) R is an RD-ring.

(iv) R is a PD-ring.

(v) R is a self-cogenerator ring.

In [19], Ikeda and Nakayama investigated the relationship between certain conditions on right and left annihilators and R-homomorphisms between right ideals of a ring which are given by left multiplication by an element of the ring. Their results are applicable to the classes of rings under consideration here. Our next result is a modification of theorem 1(i) of [19]. (Put A = R in the following for Ikeda's and Nakayama's result).

2.2.15 LEMMA:

Let A be an ideal of a ring R and suppose $x \in R$. Then $xR \cap A = rl(x) \cap A$ if and only if every left R-homomorphism $f: Rx \to A$ is given by right multiplication by an element of R.

Proof: Suppose $xR \cap A = rl(x) \cap A$ and $f: Rx \to A$ is a left R-homomorphism. If $r \in R$ then $0 = rx \Rightarrow 0 = f(rx) = rf(x)$, so $l(x) \leq l(f(x))$. Therefore, $f(x) \in rl(f(x)) \cap A \leq rl(x) \cap A$

$$= xR \cap A,$$

so $f(x) = xz$ for some $z \in R$. Clearly f is given by right multiplication by z. Conversely, suppose every R-homomorphism $f:Rx \to A$ is given by right multiplication. If $a \in rl(x) \cap A$, then we can define an R-homomorphism $f:Rx \to A$ by $f(rx) = ra$ for each $r \in R$, so $f(x) = a$. Now for some $z \in R$, $xz = f(x) = a$, so $a \in xR$. Thus $rl(x) \cap A \subseteq xR$. Since $xR \cap A \leq rl(x) \cap A$ trivially, the result follows.

2.2.16 <u>COROLLARY:</u>

Let L be a left ideal of a right RD-ring R, and suppose $lr(L) = Rx$ for some $x \in R$. Then $L = Rx$.

<u>Proof</u>: We use the same method as in the proof of 2.2.13. Suppose $L \neq Rx$, so by Zorn's Lemma there is a left ideal K of R maximal with respect to the property $L \subseteq K \subset lr(L) = Rx$. Now a right RD-ring is a right S-ring, so by 2.2.7 there is an R-homomorphism $f:Rx \to E_1$ with kernel K, so $f(L) = 0$, $f(x) \neq 0$. But $E_1 = r(J)$ and $xR \cap r(J) = rl(x) \cap r(J)$, so by 2.2.15 f is given by right multiplication by an element $z \in R$. Thus $Lz = 0$, so $z \in r(L) = rlr(L) = r(x)$. Hence $f = 0$, a contradiction. Therefore, $L = Rx$ as required.

We will find 2.2.16 above useful in our study of RD-rings, but before continuing this study we observe the following characterization of D-rings. The proof of this theorem also uses the methods given in 2.2.13 (taken from [4]) together with techniques developed in [19].

2.2.17 THEOREM:

The following conditions on an S-ring R are equivalent:-

(i) R is a D-ring.

(ii) Every R-homomorphism from a right or left ideal of R to R with finitely generated image is given by left or right multiplication respectively.

(iii) Every R-homomorphism from a right or left ideal of R to R with simple image is given by left or right multiplication respectively.

Proof: (ii) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (i) is proved in exactly the same way as 2.2.13, so we prove (i) $\Rightarrow$ (ii). Suppose then that R is a D-ring. Suppose I_1, I_2 are right ideals of R and $f:I_1 + I_2 \longrightarrow R$ is an R-homomorphism such that both $f|_{I_1}:I_1 \longrightarrow R$ and $f|_{I_2}:I_2 \longrightarrow R$ are given by left multiplication by elements z_1, $z_2 \in R$ respectively. Now if $x \in I_1 \cap I_2$ then $z_1x = f(x) = z_2x$, so clearly $z_1 - z_2 \in l(I_1 \cap I_2) = l(I_1) + l(I_2)$. Hence $z_1 - z_2 = y_1 + y_2$ for some $y_1 \in l(I_1)$, $y_2 \in l(I_2)$. But now if $a_1 \in I_1$, $a_2 \in I_2$ then $y_1a_1 = y_2a_2 = 0$ so

$$\begin{aligned} f(a_1 + a_2) &= f(a_1) + f(a_2) \\ &= (z_1 - y_1)a_1 + (z_2 + y_2)a_2. \end{aligned}$$

But $z_1 - z_2 = y_1 + y_2$, so $z_1 - y_1 = z_2 + y_2$, whence $f(a_1 + a_2) = (z_1 - y_1)(a_1 + a_2)$. Hence f is also given by left multiplication. Now suppose I is a right ideal of R and $f:I \longrightarrow R$ is an R-homomorphism, with finitely generated image. Let K = Ker f. Now $\frac{I}{K} \cong f(I)$ is finitely generated, so there exist elements $x_1,..,x_n \in I$ with $I = x_1R + .. + x_nR + K$. Clearly $f|_K$ is given by left multiplication by $0 \in R$, and by 2.2.15 $f|_{x_iR}:x_iR \longrightarrow R$ is given by left multiplication for $1 \leq i \leq n$. Repeated application of the above now shows f is given by left multiplication. This and symmetry proves (i) $\Rightarrow$ (ii) as required.

In view of 1.3.11, theorem 2.2.17 can be considered as a generalization of 2.2.14 . Now continuous rings can also be considered as a generalization of self-injective rings, so we now investigate the part 'continuity' plays in 2.2.14 .

2.2.18 THEOREM:

A ring R is a right RD-ring if and only if R is a right continuous right PD-ring.

Proof: Suppose R is a right RD-ring, hence also a right PD-ring. Let I be a right ideal of R. Now $l(I) + J = Re + J$ for some $e = e^2 \in R$, and by 1.2.15 we may assume $e \in l(I)$. Thus $rl(I) \subseteq r(e) = (1 - e)R$. Now $E_r = E_l = r(J)$, so $I \cap r(J) = rl(I) \cap r(J) = r(l(I) + J) = (1 - e)R \cap r(J) = (1 - e)E_r$. Hence $(1 - e)E_r \subseteq I \subseteq rl(I) \subseteq (1 - e)R$. Since E_r is an essential right ideal of R it follows that $(1 - e)R$ is an essential extension of I. Now suppose e is any idempotent of R, $x \in R$ and $h: eR \to xR$ is an isomorphism. Let $y = h(e) = h(e)e = ye$. If $r \in R$, clearly $er = 0 \Leftrightarrow h(er) = 0 \Leftrightarrow yr = 0$, so $r(y) = r(e)$, whence $lr(y) = lr(e) = Re$. But now by 2.2.16, $Ry = Re$, so $e = zy$ for some $z \in R$. Let $f = yez$, an idempotent by 1.5.2 . Now $e = zy$, so clearly $fy = ye^2 = y$. Hence $fR = yezR \subseteq yR = fyR \subseteq fR$, so $fR = yR = h(e)R = h(eR) = xR$. Thus xR is idempotently generated, so R is right continuous. Conversely, suppose R is a right continuous right PD-ring. Let I be a right ideal of R and let eR be an essential extension of I where $e = e^2 \in R$. Now

$eE_r = eR \cap E_r = I \cap E_r = I \cap r(J)$. But $I \leq eR$, so $rl(I) \leq rl(e) = eR$, whence $I \cap r(J) \leq rl(I) \cap r(J) \leq eR \cap E_r$

$$= I \cap r(J).$$

Hence $I \cap r(J) = rl(I) \cap r(J)$. It follows (for example from 2.2.6) that R is a right RD-ring.

2.2.19 <u>THEOREM</u>:

A ring R is an RD-ring if and only if R is a continuous S-ring.

<u>Proof</u>: It follows from 2.2.18 that an RD-ring is a continuous S-ring, so we suppose R is a continuous S-ring and deduce that R is an RD-ring. Let L be a maximal left ideal of R. If $0 \neq z \in r(L)$, then (as shown in 2.2.7) $\frac{R}{L} \cong Rz$, so Rz is a minimal left ideal of R. Hence $r(L) \leq E_l$, so $l(E_l) \leq lr(L)$. Now $r(L) \neq 0$, so $L \leq lr(L) \neq R$ and by the maximality of L, $L = lr(L)$. Therefore $l(E_l) \leq lr(L) = L$. Thus $l(E_l)$ is contained in every maximal left ideal of R, so $l(E_l) \leq J$. Suppose I is a right ideal and $I \cap E_l = 0$. Now I has an essential extension of the form eR for some $e = e^2 \in R$, so $eR \cap E_l = eE_l = 0$. Thus $e \in l(E_l) \leq J$. But J contains no idempotents, so $e = 0$ whence $I = 0$. Hence E_l is an essential right ideal, and in particular E_l contains every minimal right ideal, so $E_r \leq E_l$. Hence by symmetry, $E_r = E_l$. We have also established that $l(E_l) \leq J$. But by Nakayama's Lemma (1.1.4) if M is a minimal left ideal of R then $JM = 0$, so $J \leq l(E_l)$ whence $J = l(E_l) = l(E_r)$. Let L be a maximal left ideal of R. $r(L) \neq 0$, so there is a minimal right ideal $M \leq r(L)$. Let e be an idempotent with eR an essential extension of M, so clearly $M = eR \cap E_r = eE_r$. Now

$z \in l(M) \Leftrightarrow zE_r = 0 \Leftrightarrow ze \in l(E_r) = J \Leftrightarrow z \in R(1 - e) + J$. But $M \leq r(L)$, so $L \leq l(M) = R(1 - e) + J \neq R$, and by the maximality of L, $L = R(1 - e) + J$. Thus every maximal left ideal of R is of the form $R(1 - e) + J$ for some idempotent $e \in R$. The same arguments applied in 1.4.3 now show that R is semi-perfect. Suppose f is a primitive idempotent of R, so $L = R(1 - f) + J$ is a maximal left ideal. Now as above, there is an idempotent $e \in R$ with eE_r a minimal right ideal and $L = l(eE_r) = R(1 - e) + J$. Then

$fE_r = r(R(1 - f) + J) = r(L) = r(R(1 - e) + J) = eE_r$,

a minimal right ideal. Since $E_r = E_l$ is an essential right ideal, it follows that fR is a uniform right ideal. This, symmetry, and 2.2.8 show that R is a PD-ring, so by 2.2.18 R is an RD-ring, as required.

Theorem 2.2.19 can be considered as a generalization of our result on self-injective RD-rings (2.2.14(i) $\Leftrightarrow$ (iii)). It is also a generalization of the following result, which was proved by Utumi in [40], theorem 7.10 .

2.2.20 <u>COROLLARY</u>:

Let R be an Artinian ring. Then R is continuous if and only if R is a Quasi-Frobenius ring.

<u>Proof</u>: Clearly by 2.2.19 it is enough to show that if R is continuous then R is an S-ring. Suppose then R is an Artinian continuous ring. Let M be a minimal right ideal of R and suppose $j \in J$ with $jM \neq 0$, whence clearly $r(j) \cap M = 0$. Now M has an essential extension fR for some idempotent $f \in R$, so $r(j) \cap fR = 0$. Therefore $fR \cong jfR$. But R is continuous,

so jfR is generated by an idempotent, and $jfR \subseteq J$ which contains no idempotents, a contradiction. Thus $JM = 0$, so $JE_r = 0$, i.e. $E_r \subseteq r(J)$. Now if L is a maximal left ideal of R then $L = R(1 - e) + J$ for some idempotent $e \in R$, and since R is Artinian $0 \neq E_r \cap eR \subseteq eR \cap r(J) = r(R(1 - e) + J)$

$$= r(L).$$

Hence R is a right S-ring. The result now follows by symmetry.

We end this section with some notation. In 2.2.4 we proved that if R is a D-ring, a right RD-ring or a right PD-ring, then $E_r = E_l$ = the sum of all minimal ideals of R. For convenience, we now introduce

2.2.21 Notation: If R is a ring in which $E_r = E_l$, then we write $E(R)$ or simply E instead of E_r or E_l. The symbol E will not be used for any other purpose, and its use will imply $E_r = E_l$.

§ 3 Properties of D-, RD-, and PD-rings.

We start this section by considering the relationship between a ring R, which is either a D-ring, an RD-ring or a PD-ring, and fRf, where f is a generating idempotent of R (i.e. R = RfR). The reader is asked to recall 1.5.13 to 1.5.16 (inclusive) in particular. In this situation, we apply our standard notation to R. Thus $J = J(R)$, the Jacobson radical of R, and if X is a subset of R then $l(X) = l_R(X)$ (even if $X \subseteq fRf$). It is straightforward to verify that $J(fRf) = fJf$.

2.3.1 PROPOSITION:

Let f be a generating idempotent of a ring R, and let I be a right ideal of fRf. Then

(i) $l_{fRf}(I) = fl(I)f = fl(IR \oplus (1 - f)R)$.

(ii) $r_{fRf}l_{fRf}(I) = rl(I)f = frl(IR \oplus (1 - f)R)f$.

Proof: Clearly IR is a right ideal of R, $IR \subseteq fR$, and $I = IRf$. Now $1 - f \in l(I)$, so $l(I) = l(I)f + R(1 - f)$ and $l(I)f = l(IR) \cap Rf = l(IR \oplus (1 - f)R)$. Now $R = RfR$, so $l(I)f = RfRl(I)f$. But $fRl(I)f \subseteq l(I)f \cap fRf = l_{fRf}(I)$, so $l(I)f \leq Rl_{fRf}(I) \leq l(I)f$. Thus $Rl_{fRf}(I) = l(I)f = l(IR \oplus (1 - f)R)$. Clearly now (i) holds, as well as a symmetrical result for right annihilators of left ideals of fRf. Now $I \subseteq fRf$ so $rl(I) \subseteq rl(f) = fR$ and $z \in r(l(I)f) \Leftrightarrow fz \in rl(I) \Leftrightarrow z \in rl(I) \oplus (1 - f)R$. Hence

$$r_{fRf}l_{fRf}(I) = fr(l_{fRf}(I))f = fr(Rl_{fRf}(I))f = fr(l(I)f)f = frl(I)f.$$

Since $rl(I) \subseteq fR$ and $l(I)f = l(IR \oplus (1 - f)R)$, the result follows.

2.3.2 THEOREM:

Let f be a generating idempotent of a ring R. Then

(i) R is a right PD-ring $\Rightarrow$ fRf is a right PD-ring.

(ii) R is a right RD-ring $\Rightarrow$ fRf is a right RD-ring.

(iii) R is a D-ring $\Rightarrow$ fRf is a D-ring.

Proof: (iii) follows immediately from 2.3.1 . Suppose R is a right PD-ring, so by 1.5.13 fRf is semi-perfect. Now $J(fRf) = fJf$ and $Jf = RfRJf = RfJf$, so by 2.3.1

$$r_{fRf}(fJf) = r(Jf \oplus R(1 - f))f = r(J + R(1 - f))f$$
$$= (E \cap fR)f = fEf.$$

If I is a right ideal of fRf then $IR \leq fR$ so clearly $(I \cap fEf)R = IR \cap E$. It follows that fEf is an essential right ideal of fRf. Also $r_{fRf}l_{fRf}(I) \cap fEf = rl(I)f \cap fEf$

$$= (rl(I) \cap E)f \text{ since}$$

$rl(I) \subseteq rl(f) = fR$. If L is a left ideal of fRf, then by 2.3.1 $l_{fRf}r_{fRf}(L) + fJf = f(lr(RL + R(1 - f)) + J)f$.

In particular, if $fJf \leq L$ then $Jf = RfJf \subseteq RL$, so $J \leq RL + R(1 - f)$. Clearly now (i) and (ii) hold.

We will see that the converses of 2.3.2(i), (ii) and (iii) do not hold. But first we establish that under two-sided conditions 2.3.2(i) does have a converse.

2.3.3 THEOREM:

Let f be a generating idempotent of a ring R. Then

(i) R is a PD-ring $\Leftrightarrow$ fRf is a PD-ring.

(ii) R is a self-injective D-ring $\Leftrightarrow$ fRf is a self-injective D-ring.

Proof: Suppose fRf is a PD-ring. As in 2.3.2 we see that $fr(J)f = r_{fRf}(fJf) = E(fRf) = l_{fRf}(fJf) = fl(J)f$, and since $RfR = R$, it follows that $r(J) = l(J)$. By 1.5.13 R is semi-perfect, so by 1.5.9 $E_r = l(J) = r(J) = E_l = E$, so $E(fRf) = fEf$. Suppose e is a primitive idempotent of fRf and I is a non-zero right ideal of R, with $I \leq eR$. Now $I = IRfR = IfR$, so $0 \neq If$. Since $e \in fRf$, $0 \neq efEf = eEf \subseteq If$, so $0 \neq eE = eEfR \subseteq IfR = I$. Hence eR is a uniform right ideal of R. If g is a primitive idempotent of R, then $gR = gRfR \not\subseteq J$, so $gRfRf = gRf \not\subseteq J$. Hence there is a primitive idempotent $e \in fRf$ with $gRe \not\subseteq J$, so by 1.5.1 $gR \cong eR$, a uniform right ideal of R with a (unique) minimal right R-submodule $eE \cong gE$. It follows from 1.3.3 that E is an essential right ideal of R. This, symmetry and 2.2.8 show R is a PD-ring, so (i) holds by 2.3.2 . (ii) follows immediately from (i), 1.5.16, and 2.2.14 .

Our next result follows immediately from a result proved by Utumi in [40], theorem 7.1, under the (weaker) hypothesis that R is a right continuous ring which is not right self-injective. For completeness, we provide a proof here which uses the stronger properties of right RD-rings.

2.3.4 THEOREM:

Let R be a right RD-ring, and suppose R is not right self-injective. Then there is a primitive idempotent $e \in R$ such that for any primitive idempotent $f \in R$, $eR \cong fR$ if and only if $eR \cap fR \neq 0$.

<u>Proof</u>: Since R is not right self-injective, by 1.3.11 there is a right ideal I of R and an R-homomorphism $h:I \longrightarrow R$ which is not given by left multiplication. Let $1 = e_1 + \dots + e_n$, a sum of mutually orthogonal primitive idempotents of R. For each i, define $e_ih:I \longrightarrow e_iR$ by $e_ih(a) = e_i(h(a))$ for all $a \in I$. Clearly $h = e_1h + \dots + e_nh$, and h is given by left multiplication if and only if each e_ih is. Hence we may assume $h(I) \leq e_iR$ for some i. We write $e = e_i$. Let $K = \{a - h(a):a \in I\}$, a right ideal of R. Suppose $l(K)e \not\leq J$, so by 1.2.10 $l(K)e = Re$. Thus $e = xe$ for some $x \in l(K)$. Now $x(a - h(a)) = 0$, that is, $xa = xh(a)$ for all $a \in I$, and $h(I) \leq eR$, so $xa = xeh(a) = eh(a) = h(a)$ for all $a \in I$, contradicting the assumption that h is not given by left multiplication. Thus $l(K)e \leq J$. Then $l(K) \leq R(1 - e) + J$, so $eE = r(R(1 - e) + J) \leq rl(K) \cap E = K \cap E$. Suppose $a \in I$, $0 \neq a - h(a) \in eE$. Then $h(a) \in eR$, so $a \in eR$, and since $0 \neq a - h(a)$, $0 \neq a \in I \cap eR$. Hence $eE \leq I$. Now $eE \not\leq I \cap (1 - e)R$, so $h|_{I \cap (1-e)R}$ is given by left multiplication, whence h can be extended to an R-homomorphism $h':I + (1 - e)R \longrightarrow eR$. Now $I + (1 - e)R = eI \oplus (1 - e)R$, and clearly $h'|_{(1-e)R}$ is given by left multiplication, but h' is not, so it follows that $h'|_{eI}$ is not given by left multiplication. Hence we may assume $I \leq eR$. Suppose f is an idempotent of R and $g:eR \longrightarrow fR$ is an isomorphism. As in 1.5.1, g and g^{-1} are given by left multiplication, so clearly $gh:I \longrightarrow fR$ is not. Hence as above, $fE \leq I \leq eR$, whence $0 \neq eE = fE \leq eR \cap fR$, as required. Conversely, if e, f are primitive idempotents and $eR \cap fR \neq 0$, then $0 \neq eE = fE$, so clearly $fe \notin l(E) = J$, whence by 1.5.1, $eR \cong fR$.

Combining 1.5.15, 2.2.14, 2.3.3, and 2.3.4, we now have:-

2.3.5 COROLLARY:

The following conditions on a ring R are equivalent:-

(i) R is a self-injective D-ring.

(ii) The nxn matrix ring R_n is a self-injective D-ring for all $n \in \mathbb{N}$.

(iii) R_n is an RD-ring for some $n \in \mathbb{N}$, $1<n$.

In 3.2.2 we give an example of a D-ring which is not self-injective, so 2.3.5 above shows that the converses to 2.3.2(ii) and (iii) do not hold. In 3.1.6 we give an example to show that the converse to 2.3.2(i) does not hold. Hence the only properties discussed in this chapter that are Morita invariant are PD-rings and self-injective D-rings.

Finally, we note that in 3.3.8 we give an example of a self-injective D-ring R which has an idempotent $e \in R$ such that eRe is not self-injective and $E_r(eRe) = E_l(eRe) = 0$. Of course $ReR \neq R$.

The following theorem was proved in the Artinian case by Ikeda in [18], theorem 7.

2.3.6 THEOREM:

Let R be a ring such that R and every factor ring of R is a right PD-ring. Then R is a right Artinian ring with primary decomposition. Further, if e is a primitive idempotent of R, then eR has a unique composition series.

Proof: Let $T = \bigcap_{n=1}^{\infty} J^n$. Suppose $T = 0$, so $\bigcap_{n=1}^{\infty} (J^n \cap E) = 0$ and by 2.1.12 $\bigcup_{n=1}^{\infty} l(J^n \cap E) = R$. But $1 \in R$ so for some $k \in \mathbb{N}$, $l(J^k \cap E) = R$, whence $J^k \cap E = 0$, and since E is an essential right ideal, $J^k = 0$. Even if $T \neq 0$ we can apply this argument to the factor ring $\frac{R}{T}$, thus obtaining $k \in \mathbb{N}$ such that $J^k = J^{k+1} = \ldots = \bigcap_{n=1}^{\infty} J^n = T$. Suppose F is an ideal of R, $r(J^k) \subseteq F$, and $\frac{F}{r(J^k)} \subseteq E\left(\frac{R}{r(J^k)}\right)$. Then $\frac{F}{r(J^k)}$ is a completely reducible left R-module, so $JF \subseteq r(J^k)$ whence $F \subseteq r(J^{k+1})$. But $J^{k+1} = J^k$, so $F \subseteq r(J^k)$ whence $E\left(\frac{R}{r(J^k)}\right) = 0$. Hence $r(J^k) = R$, so $J^k = 0$. If $t \in \mathbb{N}$, $1 \leq t \leq k$, then $\frac{R}{J^t}$ is a right PD-ring and $\frac{J^{t-1}}{J^t} \subseteq E\left(\frac{R}{J^t}\right)$, so $\frac{J^{t-1}}{J^t}$ is a finite dimensional completely reducible right R-module, so has a (right) composition series. It is now straightforward to construct a composition series for R_R, so by 1.2.4 R is right Artinian. By 1.6.10, to show R has primary decomposition it is enough to show that if A is a meet-irreducible ideal of R then A is primary. By considering the factor ring $\frac{R}{A}$ we may assume $A = 0$. Let M_1, M_2 be distinct maximal ideals of R, so $M_1 + M_2 = R$. Then $0 = r(M_1 + M_2) = r(M_1) \cap r(M_2)$. But 0 is meet-irreducible and $r(M_1) \neq 0$, $r(M_2) \neq 0$, a contradiction. Hence R has a unique maximal ideal which clearly is $J = W$. Since W is nilpotent, clearly 0 is primary as required. If R is semi-simple Artinian the final statement of the theorem is trivial. Since R is right Artinian, it follows by induction that we may assume the final statement holds in every proper factor ring of R, and in particular in $\frac{R}{E}$. But if e is a primitive idempotent of R, then eE is the unique minimal right R-submodule of eR, so the result follows easily.

Let e be an idempotent of a primary Artinian ring, so $R = ReR$. Now $J^k = 0$, $J^{k-1} \neq 0$ for some $k \in \mathbb{N}$, so $0 \neq J^{k-1} = ReRJ^{k-1}$, whence $0 \neq eJ^{k-1} \leq eE_r$. Clearly now if fR and Rf are uniform right and left ideals of R respectively for every primitive idempotent $f \in R$, then $E_r = E_l = J^{k-1}$, so (by 2.2.8) R is a Quasi-Frobenius ring.

In [1], Satz 5, Asano showed that an Artinian ring R was a principal right and left ideal ring if and only if R has primary decomposition and for each primitive idempotent $e \in R$, eR and Re have unique composition series (i.e. if and only if R is a uniserial ring). Combining this result with our comments above and with 1.6.14 and 2.3.6, we immediately get

2.3.7 COROLLARY:

The following conditions on a ring R are equivalent:-

(i) R is an Artinian p.r.i.- and p.l.i.-ring.

(ii) R and every factor ring of R is a PD-ring.

(iii) R and every factor ring of R is a Quasi-Frobenius ring.

We now turn our attention to modules over D-rings. The technique we use (unusual, but very effective) is taken from [36], where Skornjakov proved 2.3.14 for self-injective D-rings (or, by his definition, self-injective annihilator rings). We start by extending our definition of the dimension of a completely reducible module.

2.3.8 Definition: Let M be a completely reducible module over a ring R. Then by 1.1.2, $M = \sum_{i \in I} \oplus M_i$ for some collection

$\{M_i\}_{i\in I}$ of simple submodules of M. It is well known that the cardinality of I, denoted $|I|$, depends only on M and not on the collection $\{M_i\}_{i\in I}$ of simple submodules of M. We say $|I|$ is the <u>dimension</u> of M, written dim (M). If $|I|$ is not finite, then we say M is <u>infinite dimensional</u>.

<u>Notation</u>: Let X be a subset of a given set Y. Then we write $X^1 = X$ and $X^{-1} = Y \setminus X = \{y \in Y : y \notin X\}$.

2.3.9 <u>Definition</u>: Let Y be a set. A collection C of subsets of Y is said to be <u>independent</u> if for any finite number of distinct elements $X_1, .., X_n$ of C, and for any $i_1, .., i_n \in \{1, -1\}$, $X_1^{\,i_1} \cap \ldots \cap X_n^{\,i_n} \neq \emptyset$.

In [35], Sikorski proves that if X is an infinite set, then there is an independent collection C of subsets of X with $|C| > |X|$. We will need to apply this result to a countably infinite set X, and for completeness, we provide a proof here.

2.3.10 <u>LEMMA</u>:

Let X be a countably infinite set. Then there is an uncountable independent collection C of subsets of X.

<u>Proof</u>: Let $\mathbb{R}$ denote the real numbers and $\mathbb{Q}$ denote the rational numbers. For each $a \in \mathbb{R}$ and each $t \in \mathbb{N}$ define $A_a^{(t)} = \{(p_1, .., p_t) \in \mathbb{Q}^t : |a - p_i| < \frac{1}{2^t} \text{ for some } i\}$, and define $A_a = \bigcup_{t=1}^{\infty} A_a^{(t)} \subseteq \bigcup_{t=1}^{\infty} \mathbb{Q}^t$. Let $Y = \bigcup_{t=1}^{\infty} \mathbb{Q}^t$. Since $\mathbb{Q}$ is countable, so is $\mathbb{Q}^t$ for each $t \in \mathbb{N}$, whence Y is countable. Suppose

$a_1,\ldots,a_n,b_1,\ldots,b_m$ are distinct elements of R. Let $d = \min\{|a_i - b_j| : 1 \leq i \leq n,\ 1 \leq j \leq m\}$. Now we can choose $t \in \mathbb{N}$ with $t \geq n$ and $\frac{1}{2^{t-1}} < d$. Clearly now if $p \in \mathbb{Q}$, $1 \leq i \leq n$ and $|a_i - p| < \frac{1}{2^t}$, then for each j, $1 \leq j \leq m$,
$|b_j - p| + |a_i - p| \geq |a_i - b_j|$, so $|b_j - p| > \frac{1}{2^{t-1}} - \frac{1}{2^t} = \frac{1}{2^t}$.
For $i = 1,\ldots,n$ we can choose $p_i \in \mathbb{Q}$ with $a_i - \frac{1}{2^t} < p_i < a_i + \frac{1}{2^t}$, so $|a_i - p_i| < \frac{1}{2^t}$. Now $t \geq n$ (by choice), so if $n + 1 \leq i \leq t$, we put $p_i = p_1$. Clearly now $(p_1,\ldots,p_t) \in A_{a_1} \cap \ldots \cap A_{a_n}$, but $(p_1,\ldots,p_t) \notin A_{b_j}$ for $1 \leq j \leq m$. Therefore
$A^1_{a_1} \cap \ldots \cap A^1_{a_n} \cap A^{-1}_{b_1} \cap \ldots \cap A^{-1}_{b_m} \neq \emptyset$. Hence $C = \{A_a : a \in \mathbb{R}\}$ is an uncountable independent set of subsets of Y. Since Y is countable, and there is a bijection between any two countable sets, the result follows.

2.3.11 <u>THEOREM</u>:

Let R be a D-ring. Then every finitely generated right R-module is finite dimensional.

<u>Proof</u>: We start by considering the socles of factor modules of R_R. Suppose I, K are right ideals of R, $K \leq I$ and $\frac{I}{K}$ is an infinite dimensional completely reducible right R-module. Let $1 = e_1 + \ldots + e_t$, a sum of mutually orthogonal primitive idempotents of R. Now (by 1.2.12 and 2.2.9) every simple right R-module is isomorphic to e_jE for some j. It follows that $\frac{I}{K}$ contains a countably infinite direct sum $\sum_{i=1}^{\infty} \oplus \frac{x_iR + K}{K}$ of isomorphic simple right R-modules, such

that there is a primitive idempotent $e \in R$ with $\frac{x_iR + K}{K} \cong eE$ for each $i \in \mathbb{N}$. Now for each $i \in \mathbb{N}$, there is an R-epimorphism $x_iR + K \to eE$ which, by 2.2.17, is given by left multiplication. Hence for each $i \in \mathbb{N}$ there is an element $a_i \in l(K)$ with $a_ix_iR = eE$. If $n \in \mathbb{N}$, $r_1,..,r_n \in R$ and $x_1r_1 + .. + x_nr_n \in K$, then since $\sum_{i=1}^{\infty} \frac{x_iR + K}{K}$ is a direct sum, each $x_ir_i \in K$, so $a_ix_ir_i = 0$. Hence for any subset M of $\mathbb{N}$ we can define an R-homomorphism $h_M: \sum_{i=1}^{\infty} x_iR + K \to eE$ by $h_M(x_i) = a_ix_i$ whenever $i \in M$, $h_M(x_i) = 0$ whenever $i \notin M$, and $h_M(K) = 0$, extending this definition by linearity. Now by 2.2.17, each h_M is given by left multiplication by an element $b_M \in l(K)$, and since the image of h_M is $eE \leq eR$, we may assume $b_M = eb_M$. By 2.3.10 there is an uncountable independent collection C of subsets of $\mathbb{N}$, so $|C| > |\mathbb{N}|$. Suppose $M_1,..,M_n$ are distinct elements of C. Now for each j, $1 \leq j \leq n$, there is an element $t_j \in M_j \cap (M_1^{-1} \cap \ldots \cap M_{j-1}^{-1} \cap M_{j+1}^{-1} \cap \ldots \cap M_n^{-1}) \subseteq \mathbb{N}$. If $1 \leq j,k \leq n$, $j \neq k$, then $t_j \notin M_k$ so $b_{M_k} x_{t_j} = h_{M_k}(x_{t_j}) = 0$. Suppose $r_1,..,r_n \in R$ and $r_1b_{M_1} + .. + r_nb_{M_n} \in l(I)$. Then for each $j \neq k$, $1 \leq j,k \leq n$, $x_{t_j} \in I$ so $r_1b_{M_1}x_{t_j} + \ldots + r_nb_{M_n}x_{t_j} = 0$, and since $j \neq k$, $r_kb_{M_k}x_{t_j} = 0$. Hence for $1 \leq j \leq n$, $r_jb_{M_j}x_{t_j} = 0$. Suppose $r_je \notin J$ for some j, so $Rr_je = Re$. Since $b_{M_j} = eb_{M_j}$, we get $b_{M_j}x_{t_j} \in Rr_jeb_{M_j}x_{t_j} = Rr_jb_{M_j}x_{t_j} = 0$. But $t_j \in M_j$, so $b_{M_j}x_{t_j} = h_{M_j}(x_{t_j}) \neq 0$, a contradiction. Thus

$r_j b_{M_j} = r_j e b_{M_j} \in Jl(K)$ for each j. Now $\frac{I}{K}$ is a completely reducible right R-module, so (by 1.1.2 and 1.1.4) $IJ \subseteq K$. Thus $l(K)IJ = 0$, so $l(K)I \subseteq l(J) = E = r(J)$, i.e. $Jl(K)I = 0$, whence $Jl(K) \leq l(I)$. Hence $\sum_{M \in C} \frac{Rb_M + Jl(K)}{Jl(K)}$ is a direct sum. Now if $M \in C$ and $i \in M$, then $x_i \in I$ and $b_M x_i = h_M(x_i) \neq 0$, so $b_M \notin l(I)$. Hence $\sum_{M \in C} \frac{Rb_M + l(I)}{l(I)}$ is an (uncountable) direct sum of non-zero submodules of $\frac{l(K)}{l(I)}$, and since $Jl(K) \leq l(I)$, $\frac{l(K)}{l(I)}$ is a completely reducible left R-module. Thus $\dim\left(\frac{l(K)}{l(I)}\right) \geq |C| > |\mathbb{N}|$. Now $K = rl(K)$ and $I = rl(I)$, so a symmetric argument now gives $\dim\left(\frac{I}{K}\right) > |\mathbb{N}|$. But this holds whenever $\frac{I}{K}$ is an infinite dimensional completely reducible right R-module, and in particular when $\frac{I}{K} = \sum_{i=1}^{\infty} \oplus \frac{x_i R + K}{K}$, whence clearly $\dim\left(\frac{I}{K}\right) = |\mathbb{N}|$, a contradiction. Since every cyclic right R-module is isomorphic to $\frac{R_R}{K}$ for some right ideal K of R, we have established that every cyclic right R-module has finite dimensional socle. Inductively, suppose $n \in \mathbb{N}$ and every right R-module generated by at most n elements has finite dimensional socle. Let $M = m_1 R + \ldots + m_{n+1} R$ be a right R-module (generated by n+1 elements). Let $N = m_2 R + \ldots + m_{n+1} R$ so by the induction hypothesis, $E(N) = E(M) \cap N$ is finite dimensional. Now $E(M) \cap N$ is a direct summand of $E(M)$ (by 1.1.2), so $E(M) \cong (E(M) \cap N) \oplus \frac{e(M)}{E(M) \cap N}$, whence $\frac{E(M)}{E(M) \cap N}$ is a completely reducible right R-module, and since

$$\frac{E(M)}{E(M) \cap N} \cong \frac{E(M) + N}{N} \leq_R \frac{N + m_1 R}{N} \cong \frac{m_1 R}{m_1 R \cap N},$$ a cyclic R-module, $\frac{E(M)}{E(M) \cap N}$ is finite dimensional. Hence $E(M) \cong (E(M) \cap N) \oplus \frac{E(M)}{E(M) \cap N}$ is finite dimensional. Thus by

induction, every finitely generated right R-module has finite dimensional socle. Finally, suppose M is any finitely generated right R-module containing a direct sum $\sum_{x\in X} \oplus M_x$ of non-zero submodules of M. For each $x \in X$, choose $0 \neq m_x \in M_x$. Let $N = \sum_{x\in X} \oplus m_x R \subseteq M$ and let $N_o = \sum_{x\in X} m_x J \subseteq N$. Now (by 1.1.4 and 1.2.2) $\frac{m_x R}{m_x J}$ is a non-zero completely reducible right R-module, so clearly $\sum_{x\in X} \frac{m_x R + N_o}{N_o}$ is a direct sum of non-zero submodules of $E(\frac{M}{N_o})$. But M is finitely generated, whence so is $\frac{M}{N_o}$, so $E(\frac{M}{N_o})$ is finite dimensional. Thus X is a finite indexing set. Clearly now M is finite dimensional, as required.

Combining 2.3.11 with some more of Skornjakov's methods, we can now prove

2.3.12 <u>THEOREM</u>:

Let R be a D-ring, and let $T = \bigcap_{k=1}^{\infty} J^k$. Then $\frac{R}{T}$ is a Noetherian ring.

<u>Proof</u>: Let I be a right ideal of R. $\frac{R}{J}$ is Artinian, so there is a principal right ideal K_1 of R with $K_1 \subseteq I$ and $K_1 + J = I + J$. If $I \subseteq J$, we put $K_1 = 0$. Suppose inductively $n \in \mathbb{N}$ and $K_1, .., K_n$ are finitely generated right ideals, each contained in I, such that

(a) $K_i \subseteq J^{i-1}$ and $K_i \subseteq J^i \Rightarrow K_i = 0$ for $1 \leq i \leq n$

(b) $K_1 + .. + K_i + J^i = I + J^i$ for $1 \leq i \leq n$

(c) $(K_1 + \dots + K_{i-1} + J^i) \cap K_i \subseteq J^i$ for $1 \leq i \leq n$.

Now $\frac{J^n}{J^{n+1}}$ is a completely reducible (by 1.2.2) and finite dimensional (by 2.3.11 applied to $\frac{R}{J^{n+1}}$) right R-module, so (by 1.1.2) there is a finitely generated right ideal $K_{n+1} \subseteq I$ such that

$$\frac{(I + J^{n+1}) \cap J^n}{J^{n+1}} = \frac{(K_1 + \dots + K_n + J^{n+1}) \cap J^n}{J^{n+1}} \oplus \frac{K_{n+1} + J^{n+1}}{J^{n+1}}.$$

Clearly, if $K_{n+1} \leq J^{n+1}$, we can assume $K_{n+1} = 0$, so (a) above holds for $i = n + 1$. Now

$$I \cap J^n \subseteq (I + J^{n+1}) \cap J^n \subseteq K_1 + \dots + K_{n+1} + J^{n+1} \subseteq I + J^{n+1},$$

and by (b) above, $K_1 + \dots + K_n + J^n = I + J^n$, so clearly

$$I = K_1 + \dots + K_n + (I \cap J^n) \leq K_1 + \dots + K_{n+1} + J^{n+1} \subseteq I + J^{n+1}.$$

Clearly now (b) above also holds for $i = n+1$, and since $K_{n+1} \subseteq J^n$, (c) also holds for $i = n+1$. Suppose there is a sequence $\{t_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that $t_1 < t_2 < \dots$ and $K_{t_i} \neq 0$, so $K_{t_i} \nsubseteq J^{t_i}$ for each $i \in \mathbb{N}$. Choose $x_i \in K_{t_i} \subseteq J^{t_i - 1}$, $x_i \notin J^{t_i}$ for each $i \in \mathbb{N}$. Now for each $i \in \mathbb{N}$ there is a primitive idempotent $e_i \in R$ such that $x_i e_i \notin J^{t_i}$. Suppose $r_1, \dots, r_m \in R$ and $x_1 e_1 r_1 + \dots + x_m e_m r_m \in T = \bigcap_{k=1}^{\infty} J^k$. Suppose further $1 \leq i \leq m$ and $e_i r_i \notin J$, so $e_i r_i R = e_i R$, whence $x_i e_i \in x_i e_i r_i R$. Now if $j > i$ then $t_j > t_i$ so $x_j e_j \in J^{t_j - 1} \subseteq J^{t_i}$. Clearly now

$$\begin{aligned} x_i e_i &\in (x_1 e_1 r_1 R + \dots + x_{i-1} e_{i-1} r_{i-1} R + J^{t_i}) \cap x_i R \\ &\subseteq (K_1 + \dots + K_{t_i - 1} + J^{t_i}) \cap K_{t_i} \subseteq J^{t_i}, \end{aligned}$$

a contradiction. Thus $e_i r_i \in J$ for each i. But now putting $F = \sum_{i=1}^{\infty} x_i e_i J + T$, clearly $\sum_{i=1}^{\infty} \frac{x_i e_i R + F}{F}$ is an infinite direct sum of non-zero submodules of $\frac{R}{F}$, contradicting

2.3.11 . Hence for some $t \in \mathbb{N}$ we have $K_t = K_{t+1} = \ldots = 0$. Let $K = K_1 + \ldots + K_{t-1}$, a finitely generated right ideal, so clearly $I + J^n = K + J^n$ for all $n \in \mathbb{N}$. Now $T = \bigcap_{i=1}^{\infty} J^k$, so $l(T) = \bigcup_{k=1}^{\infty} l(J^k)$. Thus

$$l(I + T) = l(I) \cap \left(\bigcup_{k=1}^{\infty} l(J^k)\right) = \bigcup_{k=1}^{\infty} (l(I) \cap l(J^k)) = \bigcup_{k=1}^{\infty} l(I + J^k)$$
$$= l\left(\bigcap_{k=1}^{\infty} (I + J^k)\right),$$

so $I + T = \bigcap_{k=1}^{\infty} (I + J^k)$, and similarly $K + T = \bigcap_{k=1}^{\infty} (K + J^k)$. But now $I + T = K + T$, so $\frac{I + T}{T} = \frac{K + T}{T}$, a finitely generated right R-module. Thus every right ideal of $\frac{R}{T}$ is finitely generated, so $\frac{R}{T}$ is right Noetherian, and by symmetry, $\frac{R}{T}$ is Noetherian.

In 3.2.1 we give an example of a D-ring R in which $J = J^2 = \ldots = \bigcap_{k=1}^{\infty} J^k$, and since $\frac{R}{J}$ is Artinian, theorem 2.3.12 is, in this case, trivial. However, in 3.2.6 we give an example of a D-ring R in which (putting $T = \bigcap_{k=1}^{\infty} J^k$) $\frac{R}{T}$ is not Artinian.

2.3.13 Definition ([36]): Let I be a right ideal of a ring R. I^n is defined for each $n \in \mathbb{N}$, and we extend this definition inductively. Suppose m is an ordinal and I^n is defined for all ordinals $n < m$. If m is a limit ordinal, we define $I^m = \bigcap_{n<m} I^n$, and if m is not a limit ordinal (i.e. $m - 1$ exists) we define $I^m = II^{m-1}$. (In [36], Skornjakov says it is not hard to check that this definition is right - left symmetric (i.e. $I^m = II^{m-1} = I^{m-1}I$). Since the author has

no proof of this, we must assume here that this definition is not symmetric). If there is an ordinal n such that $I^n = 0$, then we say that I is <u>transfinitely nilpotent</u>.

Our next result was proved for self-injective D-rings by Skornjakov ([36], 'Main theorem').

2.3.14 <u>COROLLARY</u>:

Let R be a D-ring. Then J is transfinitely nilpotent if and only if R is a Quasi-Frobenius ring.

<u>Proof</u>: Let $T = \bigcap_{n=1}^{\infty} J^n$. Now $T \leq l(Jr(T)) \leq l(J^2r(T)) \leq \ldots$, so by 2.3.12, $l(J^kr(T)) = l(J^{k+1}r(T)) = \ldots$ for some $k \in \mathbb{N}$. Thus $J^kr(T) = J^{k+1}r(T) = \ldots \leq T$. Suppose inductively n is an ordinal and $J^kr(T) \leq J^m$ for all ordinals $m < n$. If $n - 1$ exists, then $J^kr(T) = J^{k+1}r(T) \leq JJ^{n-1} = J^n$, and if n is a limit ordinal, $J^kr(T) \leq \bigcap_{m<n} J^m = J^n$. Now if J is transfinitely nilpotent, induction shows that $J^kr(T) = 0$, so $J^k \leq lr(T) = T \leq J^{k+1}$. Therefore, $J^{k+1} = J^k = T$, and $J^k = J^n$ for all ordinals n. Hence $J^k = 0$, and by 1.6.3, R is Artinian. The result now follows trivially.

Let I be a right ideal of a right perfect ring R. If $I \neq 0$, then by 1.4.5 there is a maximal right R-submodule K of R, so $\frac{I}{K}$ is a simple right R-module, and by Nakayama's Lemma (1.1.4), $IJ \leq K < I$. Hence $IJ = I \Rightarrow I = 0$. Using this fact, we can now prove the following corollary to 2.3.12.

2.3.15 COROLLARY:

Let R be a D-ring. Then R is right perfect if and only if R is a Quasi-Frobenius ring.

Proof: Let $T = \bigcap_{n=1}^{\infty} J^n$. Now $T \subseteq r(l(T)J) \subseteq r(l(T)J^2) \subseteq \dots$, so by 2.3.12, $r(l(T)J^k) = r(l(T)J^{k+1}) = \dots$ for some $k \in \mathbb{N}$. Thus $l(T)J^k = l(T)J^{k+1}$, so $l(T)J^k = 0$, i.e. $J^k \subseteq rl(T) = T$. Hence $J^k = J^{k+1}$, so $J^k = 0$. But now $\frac{R}{J}$ is Artinian and $J = W$, and by 2.3.12 R is Noetherian, so by 1.2.2 R is Artinian. Hence R is a Quasi-Frobenius ring. By 1.4.5, every Artinian ring is perfect, so the converse is trivial.

In chapter 3, we will see that all our examples of D-rings can be embedded in a self-injective D-ring. It is an open question if this can always be done. We end this section with three easy results concerning subrings and overrings of D-rings.

2.3.16 LEMMA:

Let R be a subring of a ring S, and suppose R is a D-ring. Then

(i) $I = IS \cap R$ for each right ideal I of R.

(ii) $L = SL \cap R$ for each left ideal L of R.

Proof: Clearly $l_R(I) = l_R(IS \cap R)$ for each right ideal I of R, and (i) follows. (ii) follows similarly.

2.3.17 LEMMA:

Let R be a subring of a ring S, and suppose

(i) $I = (I \cap R)S$ for each right ideal I of S, and

(ii) $L = S(L \cap R)$ for each left ideal L of S.

Then if R is a D-ring, so is S.

Proof: Let I be a right ideal of R. Then

$l_S(I) = l_S((I \cap R)S) = l_S(I \cap R) = S(l_S(I \cap R) \cap R) = Sl_R(I \cap R)$.

Since an analogous relation holds for left ideals,

$r_S l_S(I) = r_R(l_S(I) \cap R)S = r_R(Sl_R(I \cap R) \cap R)S = r_R l_R(I \cap R)S$

by 2.3.16 . Hence $r_S l_S(I) = r_R l_R(I \cap R)S = (I \cap R)S = I$. The result follows by symmetry.

2.3.18 Definition: Let R be a subring of a ring S such that

(i) $I = IS \cap R$ for each right ideal I of R.

(ii) $L = SL \cap R$ for each left ideal L of R.

(iii) $I = (I \cap R)S$ for each right ideal I of S.

(iv) $L = S(L \cap R)$ for each left ideal L of S.

Then S is said to be a structure preserving overring of R.

It is easy to see that if S is a structure preserving overring of R, and T is a structure preserving overring of S, then T is a structure preserving overring of R.

2.3.19 PROPOSITION:

Let S be a structure preserving overring of a ring R. Then R is a D-ring if and only if S is a D-ring.

Proof: If R is a D-ring, then by 2.3.17, so is S. Suppose S is a D-ring and I is a right ideal of R. Then

$l_R(I) = l_S(I) \cap R = l_S(IS) \cap R$, and since an analogous relation holds for left ideals,

$r_R l_R(I) = r_S(S(l_S(IS) \cap R)) \cap R = r_S l_S(IS) \cap R = IS \cap R = I$.

The result follows by symmetry.

We note that if R is a D-ring and $c \in C(0)$, then $l(cR) = r(Rc) = 0$, so $cR = Rc = R$, whence c is a unit of R. Hence the quotient ring of a D-ring R (as described in chapter 1, section 6) is R itself.

§ 4 Group rings.

In this section, we consider group rings over D-, RD-, and PD-rings. We assume the reader is familiar with the basic theory of group rings (see, for example, [6]), but to introduce our notation, we give a brief outline here.

2.4.1 Notation: Let R be a ring, and let G be a (multiplicative) group. Then RG denotes the group ring, i.e. RG is the set of all formal sums of the form $\sum_{g\in G} a_g g$, where $a_g \in R$ for all $g \in G$, and only a finite number of the a_g's are non-zero, with addition and multiplication given as follows: for each $\sum_{g\in G} a_g g$, $\sum_{g\in G} b_g g \in RG$,

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$

$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{g\in G} (\sum_{h\in G} a_{gh^{-1}} b_h) g.$$

Note that $\sum_{h\in G} a_{gh^{-1}} b_h = \sum_{h\in G} a_h b_{h^{-1}g} \in R$. If $r \in R$, we write $r = \sum_{g\in G} a_g g$, where $a_g = 0$ for all $g \in G$, $g \neq 1_G$ (the identity of G), and $a_{1_G} = r$. Similarly, if $h \in G$, we write $h = \sum_{g\in G} a_g g$ where $a_h = 1_R$, $a_g = 0$ for all $g \in G$, $g \neq h$. (We notice that this notation involves writing $1_R = 1_G = 1_{RG}$). In this way, R and G can be considered as embedded in RG. The product of the two subsets (of RG) R and G is the whole group ring, RG, so our notation is consistent. We shall often use notation which can be interpreted by the definition of the product of two subsets of a ring. For example, if I is a right ideal of R, then $IG = \{\sum_{g\in G} a_g g \in RG : a_g \in I \text{ for all } g \in G\}$, a right ideal of RG.

Finally, we make the convention that all our standard notation applies to R. In particular, J denotes the Jacobson radical of R, and if X is a subset of R, then $l(X) = l_R(X)$. Note that JG and J(RG) may not coincide, although our first lemma shows that if G is finite, $JG \subseteq J(RG)$.

2.4.2 LEMMA:

Let R be a ring and let G be a finite group. Then $JG \subseteq J(RG)$. Further, $\frac{R}{J}$ is Artinian if and only if $\frac{RG}{J(RG)}$ is Artinian.

Proof: Let M be a maximal right ideal of RG. G is finite, so $\frac{RG}{M}$ is a finitely generated right R-module, and by Nakayama's Lemma (1.1.4), $\frac{M + JG}{M} = \frac{RG}{M}J \subset \frac{RG}{M}$. But M + JG is a right ideal of RG, so by the maximality of M, M = M + JG, i.e. $JG \subseteq M$. Hence $JG \subseteq J(RG)$. Then $\frac{RG}{J(RG)}$ is a finitely generated right $\frac{R}{J}$ - module, so if $\frac{R}{J}$ is Artinian, clearly $\frac{RG}{J(RG)}$ is an Artinian right $\frac{R}{J}$ - module, and hence is a semi-simple Artinian ring. Finally, suppose $\frac{RG}{J(RG)}$ is Artinian. Let $f:RG \rightarrow \frac{R}{J}$ be the ring epimorphism defined by $f(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g + J$ for all $\sum_{g\in G} a_g g \in RG$. Let K be the kernel of f. Now $\frac{RG}{K} \cong \frac{R}{J}$, a semi-simple ring, so $J(RG) \subseteq K$. But $\frac{RG}{J(RG)}$ is Artinian, so $\frac{R}{J} \cong \frac{RG}{K}$ is Artinian as required.

Let RG be a group ring. It is natural to ask what conditions on R and G are necessary and sufficient to ensure that RG is a D-, RD-, or PD-ring. The results in this section form only a partial answer to this question. To illustrate the problems we face, in 3.2.7 we give an example of a D-ring

R and a finite cyclic group G such that RG is not semi-perfect, so is not even a PD-ring. (We refer the reader to [41] for a discussion of when group rings are semi-perfect).

We start by showing that we can restrict our attention to finite groups, and then answer the question above in the self-injective case.

2.4.3 LEMMA:

Let R be a ring and G a group. Then RG is a right S-ring if and only if R is a right S-ring and G is a finite group.

Proof: Suppose RG is a right S-ring. If L is a proper left ideal of R, then LG is a proper left ideal of RG, so $0 \neq r_{RG}(LG) = r(L)G$, and thus $r(L) \neq 0$. Hence R is a right S-ring. Let $A = \{\sum_{g\in G} a_g g \in RG: \sum_{g\in G} a_g = 0\}$, a proper left ideal of RG. Then $r_{RG}(A) \neq 0$. Suppose $0 \neq \sum_{g\in G} b_g g \in r_{RG}(A)$. Now for all $h \in G$, $1 - h \in A$, so $0 = (1 - h)\sum_{g\in G} b_g g = \sum_{g\in G}(b_g - b_{h^{-1}g})g$, whence $b_g = b_{h^{-1}g}$ for all $g, h \in G$. Thus, $G = \{g \in G: b_g \neq 0\}$, a finite set. Conversely, suppose G is finite and R is a right S-ring. Let L be a maximal left ideal of RG. ${}_R RG$ is finitely generated, so by Zorn's Lemma, L is contained in a maximal left R-submodule M of RG. By 2.2.7, there is an R-homomorphism $f: RG \longrightarrow R$ with kernel M. Define an R-homomorphism $f': RG \longrightarrow RG$ by $f'(x) = \sum_{g\in G} f(g^{-1}x)g$ for all $x \in RG$. Clearly $f' \neq 0$ since $f \neq 0$. If $x \in L$, $g^{-1}x \in L \subseteq M$ for all $g \in G$, so $f'(x) = 0$. Suppose $h \in G$. Then

$$f'(hx) = \sum_{g\in G} f(g^{-1}hx)g = \sum_{g\in G} f((h^{-1}g)^{-1}x)g = \sum_{g\in G} f(g^{-1}x)hg$$
$$= h\sum_{g\in G} f(g^{-1}x)g = hf'(x),$$

so f' is a non-zero RG-homomorphism, and $f'(L) = 0$. By the maximality of L, clearly $L = \text{Ker } f'$, so $\frac{RG}{L} \cong \text{Im } f'$. Hence by 2.2.7, RG is a right S-ring.

2.4.4 THEOREM ([5], theorem 4.1):

Let R be a ring and let G be a finite group. Then R is right self-injective if and only if RG is right self-injective.

Proof: Let A, B be right RG-modules, and let $f:A \to B$ be an RG-monomorphism and $t:A \to RG$ an RG-homomorphism. Clearly $t':A \to R$, defined for each $a \in A$ by $t'(a) = b_1$ whenever $t(a) = \sum_{g\in G} b_g g$, is an R-homomorphism. Suppose R is right self-injective, so there is an R-homomorphism $v':B \to R$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "t'"'] & B \arrow[dl, "v'"] \\ & R & \end{tikzcd}$$

commutes, i.e. $v'f = t'$.

We define an R-homomorphism $v:B \to RG$ by $v(b) = \sum_{g\in G} v'(bg^{-1})g$ for all $b \in B$. Now if $g, \bar{g}, h \in G$ then $gh = \bar{g} \Leftrightarrow g^{-1} = h\bar{g}^{-1}$. Hence if $h \in G$, $b \in B$ then

$$v(bh) = \sum_{\bar{g}\in G} v'(bh\bar{g}^{-1})\bar{g} = \sum_{g\in G} v'(bg^{-1})gh = v(b)h.$$

Thus v is an RG-homomorphism. Now for each $a \in A$,

$$vf(a) = \sum_{g\in G} v'(f(a)g^{-1})g = \sum_{g\in G} (v'f(ag^{-1}))g = \sum_{g\in G} t'(ag^{-1})g = t(a).$$

Thus $vf = t$, so the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "t"'] & B \arrow[dl, "v"] \\ & RG & \end{tikzcd}$$

commutes.

Hence RG is right self-injective. Conversely, suppose RG is right self-injective and $f: I \longrightarrow R$ is an R-homomorphism, where I is a right ideal of R. Define $f': IG \longrightarrow RG$ by $f'(\sum_{g\in G} a_g g) = \sum_{g\in G} f(a_g)g$ for all $\sum_{g\in G} a_g g \in IG$. Clearly f' is an RG-homomorphism, so by 1.3.11 f' is given by left multiplication by an element $\sum_{g\in G} x_g g \in RG$. But now if $a \in I$, then $f'(a) = f(a) = \sum_{g\in G} (x_g a)g$, so $f(a) = x_1 a$ (and $x_g a = 0$ for $g \neq 1$). It follows from 1.3.11 that R is self-injective, as required.

If R is an Artinian ring and G a finite group, then RG is a finitely generated, hence Artinian right and left R-module, so clearly RG is an Artinian ring. Combining this fact, 2.1.3, 2.2.14 and 2.4.3, we immediately get the following corollaries to 2.4.4 .

2.4.5 <u>COROLLARY:</u>

A group ring RG is a Quasi-Frobenius ring if and only if R is a Quasi-Frobenius ring and G is a finite group.

2.4.6 <u>COROLLARY:</u>

A group ring RG is a self-injective D-ring if and only if R is a self-injective D-ring and G is a finite group.

2.4.7 <u>THEOREM:</u>

Let R be a semi-perfect ring, G a finite group, and suppose RG is also semi-perfect. Then R is a PD-ring if and only if RG is a PD-ring.

Proof: Suppose R is a PD-ring. E is an essential right ideal of R and E_R is an Artinian right R-module, so clearly EG is an essential right R-submodule of RG (see 1.3.3) and since G is finite, $(EG)_R$ is an Artinian right R-module. Clearly now EG is an essential right ideal of R (so $E_r(RG) \subseteq EG$) and is an essential extension of $E_r(RG)$. Hence $E_r(RG)$ is an essential right ideal of RG. Let $T = \frac{R}{J} \times E$ (as an abelian group). We make T a ring by defining $(a + J,x)(b + J,y) = (ab + J,ay + xb)$ for all $a,b \in R$, $x,y \in E$. (We shall use a similar construction several times in this proof). Clearly T is an Artinian ring, and it is easy to see (either direct from the definition 2.1.9 or by 2.2.8 or 2.2.9) that T is a PD-ring. Hence T is a Quasi-Frobenius ring. Now $TG \cong \frac{RG}{JG} \times EG$ is a Quasi-Frobenius ring by 2.4.5, so clearly now $E_r(RG) = E_l(RG)$ $(= E(RG))$ and $\frac{TG}{J(TG)} \times E(TG) \cong \frac{RG}{J(RG)} \times E(RG)$ is a Quasi-Frobenius ring. Since $E(RG)$ is an essential right ideal of RG, and by symmetry also an essential left ideal of RG, it is straightforward to show (either directly or using 2.2.8 or 2.2.9) that RG is a PD-ring as required. Conversely, suppose RG is a PD-ring. Let M be a minimal right ideal of R, and let I be a minimal right ideal of RG with $I \subseteq MG$. Now by 2.4.2, $JGI \subseteq J(RG)E(RG) = 0$, so $JI = 0$. Clearly the set of all coefficients (of elements of G) occurring in elements of I forms a non-zero right ideal of R contained in M, so, by the minimality of M, this right ideal must be M. But $JI = 0$, so $M \subseteq r(J) = E_l$. Thus $E_r \subseteq E_l$, and by symmetry, $E_r = E_l$ $(= E)$. Now $JG \subseteq J(RG)$ by 2.4.2 so $EG = r(J)G = r_{RG}(JG) \supseteq r_{RG}(J(RG)) = E(RG)$, an essential right and left ideal of RG. It follows that E is an essential right and left ideal of R. Now $EG \supseteq E(RG)$, so

$l(E)G = l_{RG}(EG) \subseteq l_{RG}(E(RG)) = J(RG)$, which contains no idempotents, so clearly $l(E) \subseteq J$. Therefore, $l(E) = J$, and similarly $r(E) = J$. But now $l_{RG}(EG) = r_{RG}(EG) = JG$, so it is straightforward to show that $\frac{RG}{JG} \times EG$ is an Artinian PD-ring, i.e. a Quasi-Frobenius ring. But clearly $\frac{RG}{JG} \times EG \cong (\frac{R}{J} \times E)G$, so by 2.4.5 $\frac{R}{J} \times E$ is a Quasi-Frobenius ring. It follows easily that R is a PD-ring, as required.

2.4.8 Remarks: Let G be a finite non-abelian group, and let F be an algebraically closed field of characteristic zero. Then $FG = F_{n_1} \oplus F_{n_2} \oplus .. \oplus F_{n_t}$ for some $n_1, n_2, .., n_t \in \mathbb{N}$ (see [6]), and since G is non-abelian, FG is non-commutative, so $n_i > 1$ for some i. Suppose R is a ring, $\frac{R}{J} \cong F$, and the centre of R contains an isomorphic copy of F (ensuring, by counting mutually orthogonal idempotents, that R is semi-perfect). Then a standard argument on idempotents (c.f. 4.1.8) shows that $RG = R_{n_1} \oplus \ldots \oplus R_{n_t}$. In 3.1.5 we show how to construct a right PD-ring with the above properties such that R_n is not a right PD-ring for any $n > 1$ (see 3.1.6). It follows that RG is not a right PD-ring, so theorem 2.4.7 cannot be generalized to right PD-rings. In 3.2.2 we show how to construct a D-ring R with the above properties which is not self-injective. It follows from 2.3.5 and the above that although RG is semi-perfect, RG is not even an RD-ring. Our next theorem is prompted by examination of this fact. In preparation, we make the following standard definition.

2.4.9 Definition: A finite group G is said to be Hamiltonian if every subgroup of G is normal.

We also need the following technical lemma:

2.4.10 <u>LEMMA</u>:

Let G be a finite group such that for any subgroup H of G, and for each $g \in G$, we have

$$H = x^{-1}(xHx^{-1}H)^t x \cap (xHx^{-1}H)^t \text{ for all } t \in \mathbb{N}.$$

Then G is a Hamiltonian group.

<u>Proof</u>: Since G is finite, if G is not Hamiltonian then we can choose a subgroup H of G maximal with respect to the property that H is not a normal subgroup of G. Suppose then H is such a subgroup, so for some $x \in G$, $xHx^{-1} \nsubseteq H$. Let K be the subgroup of G generated by $H \cup xHx^{-1}$. $H \subset K$, so by the choice of H, K is a normal subgroup of G. Clearly for all $t \in \mathbb{N}$, $(xHx^{-1}H)^t \leq (xHx^{-1}H)^{t+1} \leq K$. But any element of K is in $(xHx^{-1}H)^t$ for some $t \in \mathbb{N}$. Since G is finite, it follows that for some $n \in \mathbb{N}$, $K = (xHx^{-1}H)^n$. By hypothesis, $H = x^{-1}Kx \cap K$, but K is normal, so $K = x^{-1}Kx \cap K = H$, a contradiction. Hence G is Hamiltonian.

2.4.11 <u>THEOREM</u>:

Let R be a ring and G a group such that RG is a D-ring.

Then (i) G is a finite group.

(ii) R is a D-ring.

(iii) either (a) R is self-injective

or (b) G is Hamiltonian.

<u>Proof</u>: G is a finite group by 2.4.3 . Suppose I is a right ideal of R. Then $l_{RG}(IG) = l(I)G$, so $IG = r_{RG}l_{RG}(IG) = rl(I)G$, so clearly $I = rl(I)$. Similarly, if L is a left ideal of R,

then $L = lr(L)$. Hence R is a D-ring. To complete the proof, we assume R is not self-injective and deduce that G is Hamiltonian. Without loss of generality, suppose R is not right self-injective, so by 1.3.11 there is a right ideal I of R and an R-homomorphism $f:I \longrightarrow R$ which is not given by left multiplication. We aim to use the preceding lemma, so let H be any subgroup of G and $x \in G$. Let $n = |G|$ and $k = \frac{|G|}{|H|}$, and suppose $\{g_1,..,g_k\}$ is a complete set of right coset representatives of H in G. Define $v:G \longrightarrow \{1,..,k\}$ by $v(g) = i \Leftrightarrow g \in Hg_i$ for all $g \in G$. Define

$$K = \Big\{ \sum_{g\in G} (a_{v(g)} + f(a_{v(x^{-1}g)}))g\colon a_1,..a_k \in I\Big\}.$$

Suppose $a_1,..,a_k \in I$ and $y \in G$. For $1 \leq i \leq k$, let $b_i = a_j \Leftrightarrow g_i y^{-1} \in Hg_j$. Now if $g \in G$ and $v(g) = i$, then $g \in Hg_i$, so $gy^{-1} \in Hg_i y^{-1} = Hg_j$ say, whence $b_{v(g)} = b_i = a_j = a_{v(gy^{-1})}$. Therefore,

$$\begin{aligned}\sum_{g\in G} (a_{v(g)} + f(a_{v(x^{-1}g)}))gy &= \sum_{g\in G} (a_{v(gy^{-1})} + f(a_{v(x^{-1}gy^{-1})}))g \\ &= \sum_{g\in G} (b_{v(g)} + f(b_{v(x^{-1}g)}))g.\end{aligned}$$

Hence $KG \subseteq K$. But f is an R-homomorphism, so clearly $KR \subseteq K$. Thus $KRG \subseteq KG \subseteq K$, so K is a right ideal of RG. We note that if $a \in I$, then putting $a_{v(1)} = a$ and $a_{v(g)} = 0$ whenever $v(g) \neq v(1)$, we have

$$\sum_{h\in H} ah + \sum_{h\in H} f(a)xh = \sum_{g\in G} (a_{v(g)} + f(a_{v(x^{-1}g)}))g \in K.$$

(In fact K is generated by the set of all such elements). Let L be the set of all elements of the form $\sum_{y\in gHx^{-1}} b_y \in R$, where $g \in G$ and there is an element $\sum_{y\in G} c_y y \in l_{RG}(K)$ with $c_y = b_y$ for all $y \in gHx^{-1}$. Clearly L is a left ideal of R.

Suppose $1 \in L$, so there is an element $\sum_{y \in G} c_y y \in l_{RG}(K)$ and an element $g \in G$ with $\sum_{y \in gHx^{-1}} c_y = 1$. Now for all $a \in I$,

$$0 = l_{RG}(K)K \ni \left(\sum_{y \in G} c_y y\right)\left(\sum_{h \in H} ah + \sum_{h \in H} f(a)xh\right)$$

$$= \sum_{z \in G}\left(\sum_{y \in zH} c_y a + \sum_{y \in zHx^{-1}} c_y f(a)\right)z,$$

and in particular, $0 = \sum_{y \in gH} c_y a + \sum_{y \in gHx^{-1}} c_y f(a)$. But $\sum_{y \in gHx^{-1}} c_y = 1$, so $f(a) = \left(-\sum_{y \in gH} c_y\right)a$ for all $a \in I$, contradicting the choice of f (not given by left multiplication). Hence $L \neq R$, so $r(L) \neq 0$. Let

$$X = \left\{\sum_{g \in G} c_g g \in RG : \sum_{y \in gHx^{-1}} c_y \in L \text{ for all } g \in G\right\}.$$

(In fact, X is a left ideal of RG). By the definition of L, $l_{RG}(K) \subseteq X$, so $r_{RG}(X) \subseteq r_{RG}l_{RG}(K) = K$. Define

$$Y = \left\{\sum_{g \in G} b_g g \in r(L)G : yz^{-1} \in xHx^{-1},\ y,z \in G \Rightarrow b_y = b_z\right\}.$$

We will show $XY = 0$, so $Y \subseteq r_{RG}(X) \subseteq K$. (In fact, $Y = r_{RG}(X)$, a right ideal of RG). Suppose $\sum_{g \in G} c_g g \in X$ and $\sum_{g \in G} b_g g \in Y$.

Let $z_1, \ldots, z_k$ be a complete set of left coset representatives of H in G. Then $G = z_1Hx^{-1} \cup \ldots \cup z_kHx^{-1}$, and if $1 \leq i,j \leq k$, $i \neq j$, then $z_iHx^{-1} \cap z_jHx^{-1} = \emptyset$. Suppose $g \in G$ and $y = z_ihx^{-1} \in z_iHx^{-1}$ for some i, $1 \leq i \leq k$. Then

$$(y^{-1}g)(xz_i^{-1}g)^{-1} = xh^{-1}z_i^{-1}gg^{-1}z_ix^{-1} = xh^{-1}x^{-1} \in xHx^{-1},$$

so $b_{(y^{-1}g)} = b_{(xz_i^{-1}g)}$. Hence for all $g \in G$

$$\sum_{y \in G} c_y b_{(y^{-1}g)} = \sum_{i=1}^{k} \sum_{y \in z_iHx^{-1}} c_y b_{(y^{-1}g)} = \sum_{i=1}^{k}\left(\sum_{y \in z_iHx^{-1}} c_y\right) b_{(xz_i^{-1}g)}$$

$$\in Lr(L) = 0.$$

Thus $(\sum_{g\in G} c_g g)(\sum_{g\in G} b_g g) = \sum_{g\in G}(\sum_{y\in G} c_y b_{(y^{-1}g)})g = 0$, so $XY = 0$ and $Y \subseteq r_{RG}(X) \subseteq K$ as required. We define an equivalence relation $\sim$ on G as follows: for $y,z \in G$, $y\sim z$ if and only if $a_{v(y)} = a_{v(z)}$ whenever $a_1,\ldots,a_k \in I$ and $\sum_{g\in G}(a_{v(g)} + f(a_{v(x^{-1}g)}))g \in Y \subseteq K$. (Recall the definition of K). If $A,B \subseteq G$, $g \in G$, we write $g\sim B \Leftrightarrow g\sim b$ for all $b \in B$, and $A\sim B \Leftrightarrow a\sim B$ for all $a \in A$. If $g \in G$, $h \in H$, then clearly $Hg = Hhg$, so $v(g) = v(hg)$, and hence $g\sim Hg$. Suppose $a_1,\ldots,a_k \in I$ and $\sum_{g\in G}(a_{v(g)} + f(a_{v(x^{-1}g)}))g \in Y$. Suppose $y, z \in G$, $h \in H$, and $yz^{-1} = xhx^{-1} \in xHx^{-1}$. Now $x^{-1}y = hx^{-1}z$, so $v(x^{-1}y) = v(x^{-1}z)$, and $f(a_{v(x^{-1}y)}) = f(a_{v(x^{-1}z)})$. But by the definition of Y,

$a_{v(y)} + f(a_{v(x^{-1}y)}) = a_{v(z)} + f(a_{v(x^{-1}z)})$, so $a_{v(y)} = a_{v(z)}$, i.e. $y\sim z$. Now if $g \in G$, $h \in H$ then $g(xhx^{-1}g)^{-1} \in xHx^{-1}$, so $g\sim xhx^{-1}g$. Hence, for all $g \in G$, $g\sim Hg$ and $g\sim xHx^{-1}g$, so we get the (transitive) chain $g\sim Hg\sim xHx^{-1}Hg\sim HxHx^{-1}Hg\sim\ldots$. Hence for all $g \in G$ and for all $t \in \mathbb{N}$, $g\sim(xHx^{-1}H)^t g$. Suppose $y \in x^{-1}(xHx^{-1}H)^t x \cap (xHx^{-1}H)^t$ for some $t \in \mathbb{N}$. Then $1\sim(xHx^{-1}H)^t \ni y$, so $1\sim y$, and $x\sim(xHx^{-1}H)^t x \ni xy$, so $x\sim xy$. Recall that $r(L) \neq 0$. Suppose $0 \neq b \in r(L)$. Clearly $\sum_{g\in xH} bg \in Y \subseteq K$, so there are elements $a_1,\ldots,a_k \in I$ with $\sum_{g\in xH} bg = \sum_{g\in G}(a_{v(g)} + f(a_{v(x^{-1}g)}))g$. Thus $g \in xH \Rightarrow$ $b = a_{v(g)} + f(a_{v(x^{-1}g)})$, and $g \notin xH \Rightarrow 0 = a_{v(g)} + f(a_{v(x^{-1}g)})$. But $1\sim y$ and $x\sim xy$, so

$$0 \neq b = a_{v(x)} + f(a_{v(1)})$$
$$= a_{v(xy)} + f(a_{v(y)}),$$

whence $xy \in xH$, so $y \in H$. Hence $x^{-1}(xHx^{-1}H)^t x \cap (xHx^{-1}H)^t \subseteq H$. But as $1 \in H$, so $xHx^{-1} \subseteq (xHx^{-1}H)^t$ and $H \subseteq (xHx^{-1}H)^t$.

Therefore, $H \leq x^{-1}(xHx^{-1}H)^t x \cap (xHx^{-1}H)^t$, and the result now follows from 2.4.10 .

Our previous remarks (2.4.8) show that if R is a D-ring and G a finite Hamiltonian group, then RG may not be a D-ring even if RG is semi-perfect. Further, in 3.2.7 we give an example to show that even when G is a finite cyclic group and R a D-ring, RG may not be semi-perfect. However, in 3.2.4 we give an example of a D-ring R and a finite, non-abelian, Hamiltonian group G such that RG is also a D-ring.

2.4.12 PROPOSITION:

Let R be a ring and G a group such that RG is a right RD-ring. Then G is finite and R is a right RD-ring.

Proof: By 2.4.3, G is a finite group and R is a right S-ring, and by 2.4.2, $\frac{R}{J}$ is Artinian and $JG \leq J(RG)$. If L is a left ideal of R, then by 2.2.6, $lr(L) + J = L + J$. Now $E_rG = l(J)G = l_{RG}(JG) \supseteq l_{RG}(J(RG)) = E(RG)$, an essential right and left ideal of RG, so clearly E_r is an essential right and left ideal of R, whence $E_l \leq E_r$. Similarly $E_r \leq E_l$, so $E_r = E_l$ (= E). Suppose I is a right ideal of R and M is a minimal right ideal, $M \leq rl(I)$. Then

$MG \leq rl(I)G = r_{RG}l_{RG}(IG)$, so

$0 \neq MG \cap E(RG) \leq r_{RG}l_{RG}(IG) \cap E(RG) = IG \cap E(RG)$. Thus

$0 \neq MG \cap IG$, so $0 \neq M \cap I$ and by the minimality of M, $M \leq I$. So $rl(I) \cap r(J) = rl(I) \cap E = I \cap E = I \cap r(J)$. Hence R is a right RD-ring as required.

Let R be a ring and G a group such that RG is a right RD-ring. In view of theorem 2.4.11, we conjecture that either R is right self-injective or G is a finite Hamiltonian group. If $E(RG) = EG$ (i.e. $J(RG) = JG$), we can establish such a result by a simple adaptation of the proof of 2.4.11 . However, the author has no proof for the more general case. We may also suspect that theorem 2.4.11 is really a theorem on continuous rings, which suggests the following conjecture:-

2.4.13 Conjecture:

If R is a ring and G a finite group such that RG is (right) continuous, then R is (right) continuous and either R is (right) self-injective or G is Hamiltonian.

Chapter 3.

EXAMPLES.

In this chapter we will give examples of the classes of rings discussed in chapter 2. There are many examples of Quasi-Frobenius rings in the literature (see, for example, [16] and [28]), so we content ourselves here by observing that 2.3.5 (applied to Artinian rings), 2.3.7 and 2.4.5 can provide several examples of such rings.

§ 1 Preliminaries.

3.1.1 Notation: Throughout this chapter, $\mathbb{Z}$ will denote the ring of (rational) integers, a subring of $\mathbb{Q}$, the field of rational numbers. If p is a prime, then $\mathbb{Z}_{(p)}$ denotes the integers localised at $p\mathbb{Z}$, i.e. $\mathbb{Z}_{(p)} = \left\{\frac{a}{b} \in \mathbb{Q}: p \nmid b\right\}$.

We start this section by observing a standard method of constructing a ring from a given ring R, and an R - R bimodule M (c.f. the proof of 2.4.7).

3.1.2 LEMMA:

Let R be a ring and M an R - R bimodule. Let T = R x M (as an abelian group). Then T can be made into a ring by defining multiplication by

$$(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2) \text{ for all } r_1,r_2 \in R,\ m_1,m_2 \in M.$$

The proof of this lemma is straightforward and omitted. Throughout this chapter, if R is a ring and M an R - R bimodule, then R x M denotes the ring described in 3.1.2 .

We will use this construction frequently.

In many of the following examples, we will consider several rings in order to construct a specific ring. For convenience, we will then indicate that one (and only one) of the rings under consideration has standard notation applied to it. Thus if R, S, T are rings, and we say 'apply standard notation to R', then (for example) the Jacobson radicals of R, S, T will be denoted by J, J(S), J(T) respectively.

In 2.2.2, we proved that a right RD-ring is semi-perfect. Our first two examples show that if R is a ring satisfying the hypotheses of theorem 2.2.1, and either of the conditions described in the definition of a right RD-ring (2.1.8(i) or (ii)), then R may not be semi-perfect. Hence none of these conditions can be dispensed with in the proof of theorem 2.2.2 .

3.1.3 Example: Let $M = \sum_{n=0}^{\infty} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}$, a $\mathbb{Z}$ - $\mathbb{Z}$ bimodule, and let $R = \mathbb{Z} \times M$ (see 3.1.2), a commutative ring. We apply standard notation to R, so clearly $J = J(R) = 0 \times M = r(J)$. Suppose I is an ideal of R, so $I + J = n\mathbb{Z} \times M$ for some $n \in \mathbb{N} \cup \{0\}$. Now $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a submodule of M, and clearly

$0 \times \frac{\mathbb{Z}}{n\mathbb{Z}} \leq r(I + J) \leq r(I)$, so

$lr(I) + J \leq l(0 \times \frac{\mathbb{Z}}{n\mathbb{Z}}) = n\mathbb{Z} \times M = I + J$. Hence R satisfies condition 2.1.8(i). If $n = 0$, then $I \leq r(J)$, so

$l(I) + J = l(I) = l(I \cap r(J))$. Suppose $n \neq 0$. Now

$0 \times nM = (I + J)(0 \times M) = (I + J)r(J) \leq I \cap r(J)$, so

$J \leq l(I) + J \leq l(I \cap r(J)) \leq l(0 \times nM) = 0 \times M = J$. Hence

$l(I) + J = l(I \cap r(J)) = J$, and R satisfies the hypotheses of theorem 2.2.1 . Clearly R is not semi-perfect, so by 2.2.2, R does not satisfy condition 2.1.8(ii).

3.1.4 <u>Example</u>: Let F be a field, and let $T = \prod_{i=1}^{\infty} F$ (an infinite direct product of copies of F). Let $M = \sum_{i=1}^{\infty} \oplus F$, an ideal of T. For each $i \in \mathbb{N}$, let $f_i : T \to F$ be the canonical projection of T onto the ith component of T. Let $R = T \times M$ (see 3.1.2), a commutative ring, and apply standard notation to R. Clearly $J = 0 \times M = r(J)$. Suppose I is an ideal of R. Define $X = \{i \in \mathbb{N} : f_i(t) \neq 0$ for some $(t,m) \in I\}$, and define $Y = \{i \in \mathbb{N} : f_i(m) \neq 0$ for some $(0,m) \in I\}$. If $(t,m) \in I$, then $0 \times tM = (t,m)(0 \times M) \subseteq I$, so clearly $X \subseteq Y$. It is easy to see that $I \cap r(J) = I \cap (0 \times M) = \{(0,m) \in R : f_i(m) \neq 0 \Rightarrow i \in Y\}$, so $l(I \cap r(J)) = \{(t,m) \in R : f_i(t) \neq 0 \Rightarrow i \notin Y\}$. Further, $l(I) = \{(t,m) \in R : f_i(t) = f_j(m) = 0$ for all $i \in Y, j \in X\}$. Clearly now $l(I) + J = l(I \cap r(J))$, so R satisfies the hypotheses of theorem 2.2.1 . Further, we see that $rl(I) = \{(t,m) \in R : f_i(t) \neq 0, f_j(m) \neq 0 \Rightarrow i \in X, j \in Y\}$, so clearly $rl(I) \cap r(J) = I \cap r(J)$. Hence R satisfies condition 2.1.8(ii). However, R is not semi-perfect.

We notice that if T and M are as described in example 3.1.4, then T is a continuous (in fact, self-injective) semi-simple ring, and that every ideal of T and $\frac{T}{M}$ is generated by the idempotents it contains. However, $\frac{T}{M}$ contains no primitive idempotents.

Our next example, taken from Deshpande's paper [7], shows that even in the Artinian case, right RD-rings need not be left RD-rings.

3.1.5 Example ([7], example 4.5): Let F be a field, and let F(x) denote the field of all rational functions in a (commuting) indeterminate x with coefficients in F. Define $v:F(x) \longrightarrow F(x)$ by $v(f(x)) = f(x^2)$ for all $f(x) \in F(x)$. Let $M = F(x)$ as a right F(x)-module, and make M an F(x) - F(x) bimodule by defining a product function $F(x) \times M \longrightarrow M$ by $(f(x),m) \mapsto v(f(x))m$ for all $f(x) \in F(x)$, $m \in M$. Let $R = F(x) \times M$ (see 3.1.2), and apply standard notation to R. It is easy to see that $J = 0 \times M$ is the only non-zero proper right ideal of R, and that $l(J) = r(J) = J$. Hence R is a right Artinian right RD-ring. Define $L = \{(0,v(b)) \in R:b \in F(x)\}$. Clearly L is a left ideal of R, and $r(L) = J$ so $L \subset J = lr(L)$. Hence R is not a left PD-ring. Note that $0 \neq L \subset J \subset R$ is a composition series for ${}_RR$, so R is Artinian.

Let $F(x_1,x_2,\ldots)$ be the field of all rational functions in an infinite number of (commuting) indeterminates $x_1,x_2,\ldots$ with coefficients in a field F. Define $v:F(x_1,x_2,\ldots) \longrightarrow F(x_1,x_2,\ldots)$ by $v(f(x_1,x_2,\ldots)) = f(x_1^2,x_2^2,\ldots)$. Then in an analogous way to 3.1.5, we can construct a right Artinian right RD-ring R which is not left finite dimensional.

Our next example uses the ring constructed in 3.1.5 to show that if R is a right PD-ring, $1 < n \in \mathbb{N}$, then R_n may not be a right PD-ring.

3.1.6 Example: Let F, x, M, R and L be as described in example 3.1.5, and suppose $0 \neq b \in F(x)$, so $0 \neq (0,v(b)) \in L$. Let R_2 be the 2x2 matrix ring over R. Then

$\begin{bmatrix}(0,v(b)) & 0 \\ (0,v(b)x) & 0\end{bmatrix} R_2$ is a minimal right ideal of R_2, but

$1_{R_2}\left(\begin{bmatrix}(0,v(b)) & 0 \\ (0,v(b)x) & 0\end{bmatrix}\right) = \begin{bmatrix}J(R) & J(R) \\ J(R) & J(R)\end{bmatrix}$ which is clearly not a maximal left ideal of R_2. It follows that R_2 is not a right PD-ring, so by 2.3.2, the nxn matrix ring over R, R_n, is not a right PD-ring for any $n \in \mathbb{N}$, $n>1$. In 3.1.5, we showed that R is a right PD-ring. It is straightforward to check that for each $n \in \mathbb{N}$, R_n satisfies the conditions described in 2.2.4 (i) and (ii), so the converse to theorem 2.2.4 does not hold.

In 2.2.1 we showed that if R is an RD-ring, then idempotents can be lifted over J, and in 2.2.2 deduced that RD-rings are semi-perfect, and hence are PD-rings. However, PD-rings may not satisfy the hypotheses of 2.2.1 . Our next example is of a commutative ring R which satisfies 2.1.9(i),(ii) and (iii), but in which idempotents cannot be lifted over the Jacobson radical of R. Hence this ring is not semi-perfect, and in particular is not a PD-ring.

3.1.7 <u>Example</u>: Let p, q be distinct prime numbers, and let $T = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$. Clearly $\mathbb{Z}_{(p)} + \mathbb{Z}_{(q)} = \mathbb{Q}$. Define

$$M = \frac{\mathbb{Q}}{T} = \frac{\mathbb{Z}_{(p)}}{\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}} \oplus \frac{\mathbb{Z}_{(q)}}{\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}} \cong \frac{\mathbb{Z}_{(p)} + \mathbb{Z}_{(q)}}{\mathbb{Z}_{(q)}} \oplus \frac{\mathbb{Z}_{(p)} + \mathbb{Z}_{(q)}}{\mathbb{Z}_{(p)}}$$
$$= \frac{\mathbb{Q}}{\mathbb{Z}_{(q)}} \oplus \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}},$$

so M is a T - T bimodule. Let R = T x M (see 3.1.2), a commutative ring. Now the only proper submodules of $\frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}$ are those of the form $\frac{p^{-k}\mathbb{Z}_{(p)}}{\mathbb{Z}_{(p)}}$ for some $k \in \{0\} \cup \mathbb{N}$. (In fact the (additive) group $\frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}$ is isomorphic to the (multiplicative) group C_{p^∞}, that is, the group of all p^k-th roots of unity, for all $k \in \mathbb{N}$). Hence the only proper submodules of M are

(i) $H_{\infty\infty} = M$.

(ii) $H_{k\infty} = \frac{\mathbb{Z}_{(p)}}{T} \oplus \frac{p^{-k}T}{T}$ where $k \in \{0\} \cup \mathbb{N}$.

(iii) $H_{\infty l} = \frac{q^{-l}T}{T} \oplus \frac{\mathbb{Z}_{(q)}}{T}$ where $l \in \{0\} \cup \mathbb{N}$.

(iv) $H_{kl} = \frac{q^{-l}T}{T} \oplus \frac{p^{-k}T}{T}$ where $k, l \in \{0\} \cup \mathbb{N}$.

If $0 \neq a \in T$, $m \in M$, then $a\mathbb{Q} = \mathbb{Q}$, so $aM = M$, and hence $0 \times M = (a,m)(0 \times M) \subseteq (a,m)R$. It follows that the only ideals of R are

(a) $p^k q^l T \times M$ for some $k, l \in \{0\} \cup \mathbb{N}$.

(b) $0 \times H_{kl}$ for some $k, l \in \{0,\infty\} \cup \mathbb{N}$.

We apply standard notation to R. Clearly $J = pqT \times M$, the only maximal ideals of R are $pT \times M$ and $qT \times M$, and the only minimal ideals of R are $0 \times H_{01} = r(qT \times M)$ and $0 \times H_{10} = r(pT \times M)$. Clearly R satisfies 2.1.9(i),(ii) and (iii), but idempotents cannot be lifted over J. We notice that $0 \times H_{01} = (0 \times H_{0\infty}) \cap E$, $l(0 \times H_{01}) = qT \times M$, and $l(0 \times H_{0\infty}) + J = (0 \times M) + J = J = pqT \times M$. Hence, as we could have deduced earlier, R does not satisfy the hypotheses of 2.2.1 .

§ 2 Some specific examples.

In [24], Levy gives an example of a commutative local ring which is not Noetherian, but whose proper homomorphic images are self-injective. We give Levy's example in 3.2.1 below, and will show that some of the proper homomorphic images of this example are self-injective D-rings, but are not Quasi-Frobenius rings. In preparation, we recall the following:-

Definition: A totally ordered set I is said to be well-ordered if every non-empty subset of I has a least element. Clearly I is well-ordered if and only if any chain $x_0 > x_1 > \dots$ of elements of I stops after a finite number of steps. Notice that our definition allows us to say $\emptyset$ is well-ordered.

3.2.1 Example: Let I be the set of all non-negative real numbers (with the natural total ordering). Let F be a field and x a (commuting) indeterminate over F. Define R to be the set of all formal sums of the form $\sum\limits_{i\in I} a_i x^i$ such that

(i) $a_i \in F$ for each $i \in I$, and

(ii) $\{i \in I : a_i \neq 0\}$ is well-ordered.

R is an abelian group in the natural way. This construction, and the fact that we can make R into a commutative ring, is due to Neumann, [29]. We give a brief outline here. Suppose $a = \sum\limits_{i\in I} a_i x^i$, $b = \sum\limits_{i\in I} b_i x^i \in R$, and $i_o \in I$. If $j_1, j_2, \dots \in I$, $j_1 < j_2 < \dots \leq i_o$, then $i_o - j_1 > i_o - j_2 > \dots \geq 0$. It follows that $\{j \in I : a_j b_{i_o - j} \neq 0\}$ is a finite set. We may therefore define $ab = \sum_{i\in I} \left(\sum_{\substack{j\in I\\ j\le i}} a_j b_{i-j}\right)x^i$. It is straightforward to check (see [29]) that $a + b = \sum_{i\in I}(a_i + b_i)x^i \in R$, $ab \in R$, and that R is a commutative ring. If $f \in F$, we write $f = \sum_{i\in I} a_i x^i$ where $a_o = f$ and $a_i = 0$ for each $0 \ne i \in I$. Hence we can consider F as a subring of R, writing $1_F = 1_R = x^0$. Suppose $0 \ne a \in R$. For each $n \in \mathbb{N}$, let $a^n = \sum_{i\in I} a_{ni}x^i$. Let i_o be minimal with respect to $a_{1i_o} \ne 0$, and suppose $i_o \ne 0$. Now for each $i \in \mathbb{N}$, there exists $k_i \in \mathbb{N}$ with $k_i i_o > i$, whence $n \ge k_i \Rightarrow a_{ni} = 0$. Clearly now we may (consistently) write $\sum_{n=1}^{\infty} a^n = \sum_{i\in I}\left(\sum_{n=1}^{k_i} a_{ni}\right)x^i$, and it is easy to verify $\sum_{n=1}^{\infty} a^n \in R$ (see [29]). But now we have $(1 - a)(1 + \sum_{n=1}^{\infty} a^n) = 1$, so it follows that $X = \left\{\sum_{i\in I} a_i x^i \in R : a_o = 0\right\} \subseteq J(R)$. Since $\frac{R}{X} \cong F$, we see that $X = J(R)$, and R is a local ring. Suppose $0 \ne a = \sum_{i\in I} a_i x^i$, and i_o is minimal with respect to $a_{i_o} \ne 0$. Let $b = \sum_{i\in I} a_i x^{i-i_o} \in R$. Now R is local and $b \notin J$ so $bR = R$, whence $x^{i_o}R = x^{i_o}bR = aR$. Hence the only principal non-zero ideals of R are those of the form $x^iR = A_i$ say, for some $i \in I$ (so $A_o = R$). Clearly the only other non-zero ideals of R are those of the form $\sum_{\substack{j\in I\\ i<j}} x^jR = B_i$ say, for some $i \in I$ (so $B_o = J(R)$).

Clearly $A_iA_j = A_{i+j}$ and $B_iA_j = B_{i+j} = B_iB_j$ for each $i,j \in I$. Choose $0 \ne t \in I$, and put $X = A_t$ or $X = B_t$. Let $T = \frac{R}{X}$, and apply standard notation to T. If $i \in I$, $i \le t$, let $\bar{A}_i = \frac{A_i}{X}$,

$\bar{B}_i = \dfrac{B_i + X}{X}$, so $J = J(T) = \bar{B}_o$, $T = \bar{A}_o$, $0 = \bar{B}_t$, and either $\bar{A}_t = 0$ (when $X = A_t$) or $\bar{A}_t$ is the unique minimal ideal of T. Observe that for $i \in I$, $i \leq t$,

(a) if $X = A_t$, then $r(\bar{A}_i) = \bar{A}_{t-i}$, $r(\bar{B}_i) = \bar{A}_{t-i}$.

(b) if $X = B_t$, then $r(\bar{A}_i) = \bar{B}_{t-i}$, $r(\bar{B}_i) = \bar{A}_{t-i}$.

Clearly if $X = B_t$ then T is a D-ring. Since we are assuming $t \neq 0$, it follows that T is not Artinian, hence not a Quasi-Frobenius ring. Notice that $J^2 = J = B_o$, so theorem 2.3.12 is, in this case, trivial. We proceed to show that (if $X = B_t$ or $X = A_t$ then) T is self-injective (see [24]). Suppose $k \in I$, $k \leq t$, and $f:\bar{A}_k \rightarrow T$ is a T-homomorphism. Then $f(\bar{A}_k)r(\bar{A}_k) = 0$, so $f(\bar{A}_k) \subseteq lr(\bar{A}_k) = \bar{A}_k$ (by (a) and (b) above). Since $\bar{A}_k$ is a principal ideal of T, it follows that f is given by multiplication by an element of T. Now suppose $k \in I$, $k < t$, and $f:\bar{B}_k \rightarrow T$ is a T-homomorphism. Choose an infinite sequence $t > i_1 > i_2 > .. > k$ of elements of I such that $\lim_{n \to \infty}\{i_n\} = k$. For each $n \in \mathbb{N}$, the restriction $f:\bar{A}_{i_n} \rightarrow T$ is given by multiplication by an element $a_n \in T$. Since $m, n \in \mathbb{N}$, $m > n \Rightarrow i_n > i_m \Rightarrow \bar{A}_{i_n} \subseteq \bar{A}_{i_m}$, clearly $m, n \in \mathbb{N}$, $m > n \Rightarrow a_n - a_m \in r(\bar{A}_{i_n}) \subseteq \bar{A}_{t-i_n}$. We pass back to the ring R. For each $n \in \mathbb{N}$ there is an element $b_n \in R$ with $b_n + X = a_n \in T = \frac{R}{X}$. Let $b_n = \sum_{i \in I} b_{ni}x^i$. Clearly, since f is only defined on $\bar{B}_k$ and $r(\bar{B}_k) = \bar{A}_{t-k}$, we may assume that for each $t-k \leq i \in I$, $n \in \mathbb{N}$, we have $b_{ni} = 0$. Further, if $m, n \in \mathbb{N}$ and $m > n$, then since $a_n - a_m \in \bar{A}_{t-i_n}$, we have $b_n - b_m \in A_{t-i_n} = x^{t-i_n}R$, so for each $j \in I$, $j < t - i_n$, we see that $b_{nj} = b_{mj}$. Suppose $i \in I$. If $i \geq t - k$ we put $d_i = 0$.

If $i < t - k$, then $i < t - i_n$ for some $n \in \mathbb{N}$, so we put $d_i = b_{ni}$. Let $d = \sum_{i \in I} d_i x^i$. Clearly $\{i \in I : d_i \neq 0\}$ is well-ordered, so $d \in R$. Further, the construction ensures $d - b_n \in A_{t-i_n}$ for each $n \in \mathbb{N}$. Hence, putting $a = d + X \in \frac{R}{X} = T$, $a - a_n \in \bar{A}_{t-i_n} = r(\bar{B}_{i_n})$ for each $n \in \mathbb{N}$. Since the restriction $f:\bar{A}_{i_n} \to T$ is given by multiplication by a_n, we see that the further restriction $f:\bar{B}_{i_n} \to T$ is given by multiplication by a for each $n \in \mathbb{N}$. But $\lim_{n \to \infty} \{i_n\} = k$, so clearly $\bigcup_{n=1}^{\infty} \bar{B}_{i_n} = \bar{B}_k$, so $f:\bar{B}_k \to T$ is given by multiplication by $a \in T$. It follows from 1.3.11 that T is self-injective. We have established that every proper homomorphic image of R is a commutative, self-injective, local ring. If we factor out by an infinitely generated ideal of R other than J(R), we obtain a non-Artinian self-injective D-ring. If we factor out by a finitely generated (hence principal) proper ideal of R, we obtain a self-injective local ring with zero socle, which clearly is not even a PD-ring.

3.2.2 <u>Example</u>: Let I be the set of all non-negative real numbers less than or equal to 1 ($\in \mathbb{N}$). Let F be a field and x a (commuting) indeterminate over F. Define T to be the set of all formal sums of the form $\sum_{i \in I} a_i x^i$ such that

(i) $a_i \in F$ for each $i \in I$, and

(ii) $\{i \in I : a_i \neq 0\}$ is finite.

Putting $x^k = 0$ for each real number $k > 1$, T can be made into a commutative ring by defing addition and multiplication in the natural way (see 3.2.1). In the same way as 3.2.1, we see that T has only two classes of ideals, that is,

(a) $A_i = x^i T$ for some $i \in I$.

(b) $B_i = \sum_{\substack{j \in I \\ i < j}} x^j T$ for some $i \in I$.

Clearly now, a proper factor ring of the ring R considered in 3.2.1 is a structure preserving overring of T. We apply standard notation to T, so clearly $A_0 = T$, $B_0 = J = J^2$, $B_1 = 0$ and $A_1 = E$, the unique minimal ideal of T. Further $r(A_i) = B_{1-i}$ and $r(B_i) = A_{1-i}$ for each $i \in I$. Hence (by observation or by 2.3.19) T is a D-ring. We show that T is not self-injective. For each $n \in \mathbb{N}$, let $k_n = \frac{1}{2} - \frac{1}{2^n}$, and for each $i \in I$, if $i = k_n$ for some $1 \neq n \in \mathbb{N}$, let $a_i = 1$, otherwise let $a_i = 0$. Suppose $n \in \mathbb{N}$. If $m \in \mathbb{N}$, $m > n$, then $k_m > k_n$ so $x^{k_m} A_{1-k_n} = 0$. We may therefore (consistently) define $f_n : A_{1-k_n} \longrightarrow T$ by multiplication by $\sum_{i \in I} a_i x^i$. If $m,n \in \mathbb{N}$, $m > n$, then clearly f_m is an extension of f_n, and $\bigcup_{n=1}^{\infty} A_{1-k_n} = B_{\frac{1}{2}}$, so we define $f : B_{\frac{1}{2}} \longrightarrow T$ by $f(x) = f_n(x)$ whenever $x \in A_{1-k_n}$. Clearly f is a T-homomorphism, and since $\sum_{i \in I} a_i x^i \notin T$ (because $\{i \in I : a_i \neq 0\}$ is not finite) we see that f is not given by multiplication by an element of T. Hence (by 1.3.11) T is not self-injective. In preparation for our next example, we also note that since $a_i \neq 0 \Longrightarrow i \geqslant k_2 = \frac{1}{4}$, we have $f(B_{\frac{1}{2}}) \subseteq A_{\frac{1}{4}} B_{\frac{1}{2}} = B_{\frac{3}{4}} \subseteq B_{\frac{1}{2}}$, and $f(B_{\frac{3}{4}}) \subseteq A_{\frac{1}{4}} B_{\frac{3}{4}} = B_0 = 0$, so $f^2 = 0$.

3.2.3 Example: Let T and f be as in 3.2.2, so T is a commutative, local D-ring, and there is an ideal I of T with $f:I \to T$ a T-homomorphism not given by multiplication by an element of T, and $f(I) \subseteq I$, $f^2(I) = 0$. Considering T as a $T - T$ bimodule, let $R = T \times T$ (see 3.1.2). We apply standard notation to R, so $J = J(T) \times T$ and $r(J) = E = 0 \times E(T)$. Hence R is a commutative local ring with unique minimal ideal E, and $r(J) = E$, $r(E) = J$. Clearly E is essential, so R is an RD-ring. We will show R is not a D-ring. Define $A = \{(f(x),x) \in R: x \in I\}$, an ideal of R. Suppose $(s,t) \in l(A)$. Then for each $x \in I$, $0 = (s,t)(f(x),x) = (sf(x), sx + tf(x))$. So $0 = sx + tf(x)$, i.e. $tf(x) = -sx$. Since f is not given by multiplication by an element of T, it follows that $t \in J(T)$. Clearly $s \in J(T)$, so $l(A) \subseteq J(T) \times J(T)$. Thus $E(T) \times E(T) = r(J(T) \times J(T)) \subseteq rl(A)$. In particular, if $0 \neq t \in E(T)$ then $(t,0) \subseteq rl(A)$. Since $x = 0 \Rightarrow f(x) = 0$, clearly $(t,0) \notin A$. Hence $A \neq rl(A)$, so R is not a D-ring.

3.2.4 Example: Let $F = \mathbb{Q}$, the field of rational numbers, and construct the D-ring T as described in 3.2.2 . Thus the elements of T are formal power series in an indeterminate x over $\mathbb{Q}$. Let G be the quaternion group, that is, $G = \{1,y,y^2,y^3,z,yz,y^2z,y^3z\}$ subject to the relations $y^4 = 1$, $z^4 = 1$, $y^2 = z^2$, and $zy = y^3z$. G is easily seen to be a Hamiltonian group. Now by Maschke's theorem (see [6]), $\mathbb{Q}G$ is a semi-simple Artinian ring, and clearly $\frac{TG}{J(T)G} \simeq \frac{T}{J(T)}\,G \simeq \mathbb{Q}G$, so $J(TG) \subseteq J(T)G$. Hence by 2.4.2, $J(TG) = J(T)G$. Recall that by construction, we can consider $\mathbb{Q}$ as a subring of T. Define

$$e_1 = \tfrac{1}{8}(1 + y + y^2 + y^3 + z + yz + y^2z + y^3z)$$
$$e_2 = \tfrac{1}{8}(1 + y + y^2 + y^3 - z - yz - y^2z - y^3z)$$
$$e_3 = \tfrac{1}{8}(1 - y + y^2 - y^3 + z - yz + y^2z - y^3z)$$
$$e_4 = \tfrac{1}{8}(1 - y + y^2 - y^3 - z + yz - y^2z + y^3z)$$
$$f = \tfrac{1}{2}(1 - y^2).$$

Then it is straightforward to check that e_1, e_2, e_3, e_4 and f are mutually orthogonal idempotents in the centre of TG, and that $e_iTG \simeq T$ for $i = 1,2,3,4$. Further, direct calculation shows that since $\sqrt{-1} \notin \mathbb{Q}$, f is a primitive idempotent. Let $R = fTG$. We will show that R is a D-ring, whence, since clearly $TG = e_1TG \oplus \ldots \oplus e_4TG \oplus fTG \simeq T \oplus T \oplus T \oplus T \oplus R$, it will follow that TG is also a D-ring. Now f is a primitive idempotent in $\mathbb{Q}G$ also, so $\frac{R}{J(R)} = \frac{fTG}{fJ(T)G} \simeq f\mathbb{Q}G$ is a simple Artinian ring with no idempotents other than the identity, i.e. is a division ring. Hence R is a scalar local ring. Let I be the set of non-negative real numbers less than or equal to 1 ($\in \mathbb{N}$). Now every non-zero element $t \in T$ can be expressed in the form $t = x^i t_o$ for some $t_o \in T$, $t_o \notin J(T)$, and some $i \in I$. Since G is a finite group and $J(TG) = J(T)G$, whence $J(fTG) = fJ(T)G$, it follows that every non-zero element $r = fr \in R = fTG$ can be expressed in the form $x^i fr_o$ for some $fr_o \in R$, $fr_o \notin J(R)$, and some $i \in I$. But R is a scalar local ring, so $fr_oR = R$ for any $fr_o \in R$, $fr_o \notin J(R)$. Thus the only non-zero principal right ideals of R are those of the form $x^i fR = A_i$ say, for some $i \in I$ (so $A_o = R$). Clearly the only other right ideals of R are those of the form $\sum_{\substack{j \in I \\ i < j}} x^j fR = B_i$ say, for some $i \in I$ (so $B_o = J(R)$). Since xf is in the centre of R, in the same way, we see that these are also the only left ideals of R. For each $i \in I$, clearly $r_R(A_i) = B_{1-i} = l_R(A_i)$, and $l_R(B_i) = A_{1-i} = r_R(B_i)$. Thus R is a D-ring, so TG is also a D-ring. Recall from 3.2.2 that T is not self-injective, so this example shows that theorem 2.4.11 is non-trivial.

Our next example is a similar construction to that given in 3.1.7 .

3.2.5 <u>Example</u>: Let R be a prime Noetherian semi-perfect ring such that $0 \neq J(R) = xR = Rx$ for some $x \in R$. (For example, take R to be the nxn matrix ring over $\mathbb{Z}_{(p)}$ for some prime $p \in \mathbb{N}$ and some $n \in \mathbb{N}$). Let Q be the quotient ring of R (which exists by 1.6.20). It follows from 1.6.21 that $x \in C_R(0)$, so $x^{-1} \in Q$. Since $xR = Rx = J(R)$, clearly $(J(R))^n = x^nR = Rx^n$, so $x^{-n}R = Rx^{-n}$ for each $n \in \mathbb{N}$. Let $N = \bigcup_{n=1}^{\infty} x^{-n}R = \bigcup_{n=1}^{\infty} Rx^{-n}$, an R - R sub-bimodule of Q containing R, and let $M = \frac{N}{R}$. If $r \in R$, $n \in \mathbb{N}$, then we denote the elements $x^{-n}r + R$, $rx^{-n} + R \in M$ by $\overline{x^{-n}r}$, $\overline{rx^{-n}}$ respectively. Let $T = R \times M$ (see 3.1.2), a semi-perfect ring, and apply standard notation to T. Clearly, $J = J(R) \times M$ and (by 1.5.9) $E_r = E_l = 0 \times \frac{x^{-1}R}{R}$. Write $E = E_r$. Suppose $0 \neq r \in R$, so for some $n \in \mathbb{N}$, $r \in Rx^{n-1}$, $r \notin Rx^n$, whence $rx^{-n} \in R$, $rx^{-n} \notin Rx^{-1}$. Hence if $m \in M$, $0 \neq (0,\overline{rx^{-n}}) = (r,m)(0,\overline{x^{-n}}) \subseteq (r,m)T \cap E$. If $m \in M$, then $m \in \overline{Rx^{-n}}$, $m \notin \overline{Rx^{-n+1}}$ for some $n \in \mathbb{N}$, so $0 \neq (0,m)(x^{n-1},0) \in E$. Clearly now E is an essential right ideal of T. Let e be a primitive idempotent of R. Then $\frac{eRx^{-1} + R}{R} \cong \frac{eR + Rx}{Rx} = \frac{eR + J}{J}$, a simple right R-module, so clearly $(e,0)E$ is a minimal right ideal of T. Suppose $e \in R$, $m \in M$ and $(e,m) = (e,m)^2 = (e^2,em + me)$. Then $e = e^2$, and

$(e,m)E = (e,0)E$. It follows by symmetry and 2.2.8 that T is a PD-ring.

It follows from 2.3.3 and 2.3.5 that if $1 \neq n \in \mathbb{N}$, then the nxn matrix ring over the D-ring constructed in 3.2.2 is a PD-ring which is not an RD-ring. However, the construction given above enables us to give an example of a PD-ring which is not an RD-ring, and is not an nxn matrix ring for any $1 \neq n \in \mathbb{N}$.

Let $R = \begin{bmatrix} \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \\ p\mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{bmatrix}$ as an abelian group, and define multiplication in R by

$$\begin{bmatrix} a_1 & pb_1 \\ pc_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & pb_2 \\ pc_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + pb_1c_2 & p(a_1b_2 + b_1d_2) \\ p(c_1a_2 + d_1c_2) & pc_1b_2 + d_1d_2 \end{bmatrix} .$$

It is straightforward to check that R is a prime Noetherian semi-perfect ring, and that

$$J(R) = \begin{bmatrix} p\mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \\ p\mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} R = R\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} .$$

Construct Q, M and T as described above, so T is a PD-ring. We apply standard notation to T. It is well known that we can choose a sequence $a_1, a_2, \dots \in \mathbb{N}$ such that $a_1p + a_2p^2 + \dots$ cannot be (consistently) written as an element of $\mathbb{Z}_{(p)}$. For each $n \in \mathbb{N}$, let $f_n = \begin{bmatrix} 1 & 0 \\ a_1p + \dots + a_np^n & 0 \end{bmatrix}$, an idempotent of R, and let $e_n = (f_n, 0) \in T$, an idempotent of T. Put $A_n = e_nT + J^{n+1}$, a right ideal of T. It follows from the choice of $a_1, a_2, \dots$ that $\bigcap_{n=1}^{\infty} A_n \subseteq J$. Clearly for each $n \in \mathbb{N}$ we have $rl(J^{n+1}) = J^{n+1}$ and $l(A_n) = T(1 - e_n) \cap l(J^{n+1}) = l(J^{n+1})(1 - e_n)$, so $z \in l(A_n)$ $\Leftrightarrow (1 - e_n)z \in rl(J^{n+1}) = J^{n+1} \Leftrightarrow z \in e_nT + J^{n+1} = A_n$, thus $A_n = rl(A_n)$. Hence $\bigcap_{n=1}^{\infty} rl(A_n) = r(\bigcup_{n=1}^{\infty} l(A_n)) \subseteq J$, which contains no idempotents, so if $e = e^2 \in T$ and $\bigcup_{n=1}^{\infty} l(A_n) \subseteq Te$, then $e = 1$. If T is an RD-ring, hence by 2.2.18 a continuous ring, it follows that $\bigcup_{n=1}^{\infty} l(A_n)$ is an essential left ideal, so $E \leq \bigcup_{n=1}^{\infty} l(A_n)$, whence $E \subseteq l(A_n)$ for some $n \in \mathbb{N}$. But $0 \neq Ee_n \nsubseteq EA_n$ for each $n \in \mathbb{N}$, so $E \nsubseteq l(A_n)$. Hence T is not an RD-ring. Finally, we notice that

$\frac{T}{J} \simeq \frac{R}{J(R)} \simeq \frac{\mathbb{Z}_{(p)}}{p\mathbb{Z}_{(p)}} \oplus \frac{\mathbb{Z}_{(p)}}{p\mathbb{Z}_{(p)}}$, a commutative ring, so clearly T is not an nxn matrix ring for any $1 \neq n \in \mathbb{N}$.

3.2.6 Example: Let R be a prime, scalar local, Noetherian ring such that $0 \neq J(R) = xR = Rx$ for some $x \in R$. We construct Q, M and T as in 3.2.5, and apply standard notation to T. Suppose $0 \neq r \in R$, so $r \in x^{n-1}R$, $r \notin x^{n}R$ for some $n \in \mathbb{N}$. Then $x^{-n+1}r \in R$, $x^{-n+1} \notin xR$. But R is scalar local, so $x^{-n+1}rR = R$, i.e. $rR = x^{n-1}R$. Clearly now the only right or left ideals of R are the powers of $J(R) = xR$. It follows that the only right or left R-submodules of M are those of the form $\frac{x^{-n}R}{R} = \frac{Rx^{-n}}{R}$ for some $n \in \mathbb{N}$. Suppose $n \in \mathbb{N}$, $m \in M$. Then $x^{n}M = M$ so $0 \times M = (x^{n},m)(0,M) \subseteq (x^{n},m)T$. Clearly now the only (right or left) ideals of T are

(i) $x^{n}R \times M = A_n$ say for some $n \in \{0\} \cup \mathbb{N}$

(ii) $0 \times M = W$

(iii) $0 \times \frac{x^{-n}R}{R} = B_n$ say for some $n \in \{0\} \cup \mathbb{N}$.

Clearly $A_0 = T$, $A_1 = J$, $B_0 = 0$, and $B_1 = E$, the unique minimal ideal of T. Further, $r(W) = W = l(W)$, and $r(A_n) = B_n = l(A_n)$ and $r(B_n) = A_n = l(B_n)$ for each $n \in \mathbb{N}$. Hence T is a D-ring. We now consider when this example is self-injective. Suppose $t \in T$. Then clearly $tT = Tt$. If $f: tT \to T$ is a right T-homomorphism, then $f(tT)r(tT) = 0$, so $f(tT) \subseteq lr(tT) = tT = Tt$. Thus $f(t) = xt$ for some $x \in T$, whence f is given by left multiplication. Since there is only one non-principal ideal of T, namely $W = 0 \times M$, and there is a natural one-one correspondence between right T-homomorphisms $f: 0 \times M \to T$ (where $f(W) \subseteq lr(W) = W = 0 \times M$) and right R-endomorphisms of M_R, we immediately get:-

LEMMA:

T is right self-injective if and only if every right R-endomorphism of M is given by left multiplication by an element of R.

We complete this example by putting $R = \mathbb{Z}_{(p)}$ in the above (so $J(R) = p\mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}p$), for some prime $p \in \mathbb{Z}$, and deducing that T is a commutative D-ring, but is not self-injective. Suppose $a_0, a_1, a_2, \ldots \in \mathbb{N}$. For each $i \in \mathbb{N}$ define $h_i : \frac{p^{-i}R}{R} \longrightarrow M$ by
$h_i(p^{-i}r + R) = (a_0 + a_1p + \ldots + a_{i-1}p^{i-1})p^{-i}r + R$ for each $r \in R$. Clearly h_i is an R-homomorphism, and since $p^ip^{-i} = 1 \in R$, h_i is an extension of h_j for each $i, j \in \mathbb{N}$, $i > j$. Hence we define $h : M \longrightarrow M$ by $h(m) = h_i(m)$ whenever $m \in \frac{p^{-i}R}{R}$. It only remains to choose $a_0, a_1, \ldots$ such that $a_0 + a_1p + a_2p^2 + \ldots$ cannot be (consistently) written as an element of $\mathbb{Z}_{(p)}$. (Notice that $1 + p + p^2 + \ldots$ '=' $\frac{1}{1 - p}$). It is well known that this can be done, for example, by choosing $a_n = 1$ if n is prime, and $a_n = 0$ otherwise. Hence we can find an R-homomorphism $h : M \longrightarrow M$ not given by multiplication by an element of R, so T is not self-injective. Let H be the ring of all R-endomorphisms of M (called the ring of p-adic integers), and for each $r \in R$ let h_r be the element of H given by multiplication by r. If $0 \neq h \in H$, then clearly h is an epimorphism, so $h = h_{(p^k)}h'$ for some $k \in \mathbb{N}$ and some isomorphism h' of H. Hence the only right ideals of H are those of the form $h_{(p^k)}H$ for some $k \in \mathbb{N}$. If we identify R with its image in H under the ring monomorphism $r \longrightarrow h_r$ for each $r \in R$, then H is a structure preserving overring of R.

Clearly M is an H - H bimodule, and every H-endomorphism of M is given by multiplication by an element of M. Hence H x M (see 3.1.2) is a self-injective D-ring, and is a structure preserving overring of the D-ring T.

Our last example in this section completes our observations on group rings over D-rings. This example is, in all essential details, that given by Woods in [41], page 129.

3.2.7 <u>Example</u>: Let $R = \mathbb{Z}_{(7)}$, and let $F = \frac{R}{J(R)}$, the finite field with 7 elements. Let G be the cyclic group of order 3, i.e. $G = \{1, y, y^2\}$ where $y^3 = 1$. By Maschke's theorem (see [6]), $FG = \frac{R}{J(R)} G \simeq \frac{RG}{J(R)G}$ is semi-simple Artinian, so $J(RG) \subseteq J(R)G$, and by 2.4.2, $J(RG) = J(R)G$. Let $e = 5 + 6y + 3y^2$. Clearly $e - e^2 \in J(RG)$, but we will see that e cannot be lifted over J(RG). Suppose $a + by + cy^2 = f$ is an idempotent of RG and $f - e \in J(RG)$. Then clearly $b \neq 0$, $c \neq 0$. From the relation $f = f^2$, we deduce that $a^2 + 2bc = a$, $b^2 = (1 - 2a)c$, and $c^2 = (1 - 2a)b$. So $bc = (1 - 2a)^2$, and thus $a^2 + 2(1 - 2a)^2 = a$, whence $a = \frac{1}{3}$ or $a = \frac{2}{3}$. Since $a - 5 \in J(R)$, we have $a = \frac{1}{3}$. So $b^2 = \frac{1}{3}c$, $c^2 = \frac{1}{3}b$, so $b = c = \frac{1}{3}$. But $6 - \frac{1}{3} = \frac{1}{3}(17) \notin J(R)$, so $e - f \notin J(RG)$, a contradiction. Hence e cannot be lifted over J(RG), so RG is not semi-perfect. Constructing the (not self-injective) D-ring T, as in 3.2.5 and 3.2.6, we see that TG is not semi-perfect, so is not a D-ring (in fact not even a PD-ring).

§ 3 Examples in matrix form.

Suppose e, $1 - e$ are both primitive idempotents of a D-ring R which is not self-injective, and suppose $eE = E(1 - e)$. We aim to find such a ring, so we investigate the properties of R, making simplifying assumptions where appropriate. Suppose I is a right ideal of R, $I \subseteq eR$, and $h: I \to eR$ is an R-homomorphism not given by left multiplication. (Without loss of generality, such exists, as shown in the proof of 2.3.4). Let S be the set of all idempotents $f \in R$ with $fR \cong eR$. If $f \in S$, then (see 2.3.4) the restriction $h: I \cap fR \to eR$ is also not given by left multiplication. We consider the case when $I = \bigcap_{f \in S} fR$. Now it follows from 1.5.8 that

$R(1 - e)R = \sum_{f \in S} R(1 - f)$, so $I = \bigcap_{f \in S} fR = r(R(1 - e)R)$, an ideal

of R. Now Ie is a left ideal of R, $I \subseteq eR$, so $Ie \cap E \subseteq eRe \cap E = eEe = 0$. So $Ie = 0$, and thus $I \subseteq eR(1 - e)$. Hence $eR(1 - e)$ contains I, an ideal of R which, by 2.2.17, is not finitely generated. We consider the case when $I = eR(1 - e)$, and further assume (symmetrically) that $(1 - e)Re$ is an ideal of R. Let $A = (1 - e)Re$. Then $AI + IA \subseteq I \cap A = 0$, so $IA = AI = 0$. Now we can always write R in matrix form using the ring isomorphism

$$x \mapsto \begin{bmatrix} exe & ex(1 - e) \\ (1 - e)xe & (1 - e)x(1 - e) \end{bmatrix}$$ for each $x \in R$. This

involves considering $I = eR(1 - e)$ as an $eRe - (1 - e)R(1 - e)$ bimodule, and $A = (1 - e)Re$ as a $(1 - e)R(1 - e) - eRe$ bimodule, with a 'suitable' multiplication $I \times A \to eRe$, $A \times I \to (1 - e)R(1 - e)$. However, with the assumptions made above, this multiplication is trivial. Of course these assumptions have taken us far away from the general case. However, the reader may find this an instructive introduction to the construction that follows.

Let R and T be semi-perfect rings, and let M be a non-zero R - T bimodule and N a non-zero T - R bimodule. We define $Q = \begin{bmatrix} R & M \\ N & T \end{bmatrix}$, defining addition coordinate-wise and multiplication by

$$\begin{bmatrix} r_1 & m_1 \\ n_1 & t_1 \end{bmatrix} \begin{bmatrix} r_2 & m_2 \\ n_2 & t_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & r_1m_2 + m_1t_2 \\ n_1r_2 + t_1n_2 & t_1t_2 \end{bmatrix} \quad \text{for all}$$

$r_1, r_2 \in R$, $m_1, m_2 \in M$, $n_1, n_2 \in N$, $t_1, t_2 \in T$. It is straightforward to verify that Q is a ring.

The remainder of this section is devoted to examination of the construction above, so we preserve our notation throughout.

Clearly $W(Q) \supseteq \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}$, an ideal of Q, so

$\frac{Q}{W(Q)} \simeq \frac{R}{W(R)} \oplus \frac{T}{W(T)}$. Since $W(Q) \subseteq J(Q)$, it follows that $J(Q) = \begin{bmatrix} J(R) & M \\ N & J(T) \end{bmatrix}$. Now suppose R and T are local rings, so $\frac{Q}{J(Q)}$ is a direct sum of two simple Artinian rings. It follows that if Q is a PD-ring then E(Q) is a direct sum of two minimal ideals. Since $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}$ are ideals of Q, and since the socle of a PD-ring is an essential right and left ideal, it follows that if Q is a PD-ring then neither $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$ nor $\begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$ contain any non-zero right or left ideal of Q. Suppose $r \in R$ and $rM = 0$. Then clearly $\begin{bmatrix} rR & 0 \\ 0 & 0 \end{bmatrix}$

is a right ideal of Q. We therefore make the following assumption (assumed for the remainder of this section).

3.3.1 <u>Assumption</u>: For any non-zero elements $r \in R$, $t \in T$ we have $rM \neq 0$, $Nr \neq 0$, $Mt \neq 0$, and $tN \neq 0$.

Note however that we now drop the supposition that R and T are local rings.

We aim to apply 2.2.17, so we need to investigate properties of Q-homomorphisms from right ideals of Q to minimal right ideals of Q. In preparation, we list the following conditions:-

3.3.2 <u>Conditions</u>: For any minimal submodules M_o, N_o of M_T, N_R respectively, and for any submodules R_1, T_1, M_1, N_1 of R_R, T_T, M_T, N_R respectively,

(i) Every T-homomorphism $g:T_1 \to M_o$ is given by left multiplication by an element of M.

(ii) If $a \in M$, $M_o \subseteq aT$, and $f:M_1 \to aT$ is a T-homomorphism then there are elements $r \in R$, $m \in M$ such that for each $t \in T$, $at \in M_o \Rightarrow mt = 0$ and $x \in M_1$, $f(x) = at \Rightarrow f(x) = rx + mt$. (Notice that if $a \in M_o$, then $aT \subseteq M_o$, so $m \in mT = 0$, whence f is given by left multiplication by $r \in R$).

(iii) Every R-homomorphism $g:R_1 \to N_o$ is given by left multiplication by an element of N.

(iv) If $a \in N$, $N_o \subseteq aR$, and $f:N_1 \to aR$ is an R-homomorphism then there are elements $n \in N$, $t \in T$ such that for each $r \in R$, $ar \in N_o \Rightarrow nr = 0$ and $x \in N_1$, $f(x) = ar \Rightarrow f(x) = tx + nr$.

3.3.3 LEMMA:

Every right Q-homomorphism from a right ideal of Q to a minimal right ideal of Q is given by left multiplication by an element of Q if and only if conditions 3.3.2 (i) $\rightarrow$ (iv) are satisfied.

Proof: Suppose first conditions 3.3.2(i) $\rightarrow$ (iv) are satisfied. Let I, K be right ideals of Q, K minimal, and suppose $h:I \rightarrow K$ is a right Q-homomorphism. Define $h_1:I \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} K$ by $h_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} h(x)$ for each $x \in I$, and similarly define $h_2:I \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} K$. Clearly $h = h_1 + h_2$ is given by left multiplication if and only if h_1 and h_2 are, so we may assume $h = h_1$ or $h = h_2$. Without loss of generality, suppose $h = h_1$, so $h(I) \leq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Q = \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix}$. Since K is a minimal right ideal, it follows from assumption 3.3.1 that $K = \begin{bmatrix} 0 & M_o \\ 0 & 0 \end{bmatrix}$ for some minimal right T-submodule M_o of M.

Let $I + \begin{bmatrix} 0 & 0 \\ N & T \end{bmatrix} = \begin{bmatrix} A & X \\ N & T \end{bmatrix}$, $I \cap \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_o & X_o \\ 0 & 0 \end{bmatrix}$, and $I + \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & M \\ Y & B \end{bmatrix}$, $I \cap \begin{bmatrix} 0 & 0 \\ N & T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Y_o & B_o \end{bmatrix}$. Now $h(I \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) \leq \begin{bmatrix} 0 & M_o \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$, so $I \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = I \cap \begin{bmatrix} R & 0 \\ N & 0 \end{bmatrix} \leq \operatorname{Ker} h$. If $a \in A$ then for some $y \in Y$, $\begin{bmatrix} a & 0 \\ y & 0 \end{bmatrix} \in \operatorname{Ker} h$, so $\begin{bmatrix} 0 & aM \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ y & 0 \end{bmatrix} \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \leq \operatorname{Ker} h$.

So $\begin{bmatrix} 0 & AM \\ 0 & 0 \end{bmatrix} \subseteq \text{Ker } h$. Suppose $\begin{bmatrix} 0 & 0 \\ 0 & B_o \end{bmatrix} \subseteq \text{Ker } h$. Then we can define a T-homomorphism $g:X \longrightarrow M_o$ by $g(x) = m$ whenever there is an element $b \in B$ with $\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix} \in I$ and

$h\left(\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. By 3.3.2(ii), there is an element $r \in R$ with $g(x) = rx$ for all $x \in X$. But $\begin{bmatrix} 0 & AM \\ 0 & 0 \end{bmatrix} \subseteq \text{Ker } h$, so $rAM = g(AM) = 0$, whence by 3.3.1, $r \in 1_R(A)$. Clearly now h is given by left multiplication by $\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}$. Suppose now $\begin{bmatrix} 0 & 0 \\ 0 & B_o \end{bmatrix} \nsubseteq \text{Ker } h$. We can define a T-homomorphism $g:B_o \longrightarrow M_o$ by $g(b) = m$ whenever $b \in B_o$ and $h\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$.

By 3.3.2(i) there is an element $\alpha \in M$ such that $h(b) = \alpha b$ for each $b \in B_o$. So $\alpha B_o = M_o$. Suppose $x \in X$, $b_1, b_2 \in B$, $\begin{bmatrix} 0 & x \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} 0 & x \\ 0 & b_2 \end{bmatrix} \in I$, and $h\left(\begin{bmatrix} 0 & x \\ 0 & b_1 \end{bmatrix}\right) = \begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix}$, $h\left(\begin{bmatrix} 0 & x \\ 0 & b_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & m_2 \\ 0 & 0 \end{bmatrix}$. Then $b_1 - b_2 \in B_o$, and $h\left(\begin{bmatrix} 0 & 0 \\ 0 & b_1 - b_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & m_1 - m_2 \\ 0 & 0 \end{bmatrix}$, so $m_1 - m_2 = \alpha(b_1 - b_2)$, whence $m_1 - \alpha b_1 = m_2 - \alpha b_2$. Hence we can define a T-homomorphism $f:X \longrightarrow \alpha B + M_o \subseteq \alpha T$ by $f(x) = m - \alpha b$ whenever $\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix} \in I$ and $h\left(\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. Now by 3.3.2(ii) there are elements $r \in R$, $\beta \in M$ such that $f(x) = rx + \beta b$ whenever

$f(x) = \alpha b$, and $\beta b = 0$ whenever $\alpha b \in M_o$. Suppose $\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix} \in I$

and $h\left(\begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. Then $f(x) = m - \alpha b$, and since

$m \in M_o = \alpha B_o$ there is an element $b_o \in B_o$ with $m = \alpha b_o$.

So $f(x) = \alpha(b_o - b)$. But $\alpha b_o \in M_o$ so $\beta b_o = 0$. Thus

$f(x) = \alpha(b_o - b) = rx + \beta(b_o - b) = rx - \beta b$. But

$f(x) = m - \alpha b$, so $m = rx + (\alpha - \beta)b$, that is,

$\begin{bmatrix} r & \alpha-\beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. Now $AM \subseteq X_o$, and $\begin{bmatrix} 0 & AM \\ 0 & 0 \end{bmatrix} \subseteq \text{Ker } h$

so clearly $rAM = 0$ whence by 3.3.1, $rA = 0$. It follows that

h is given by left multiplication by $\begin{bmatrix} r & \alpha-\beta \\ 0 & 0 \end{bmatrix}$ as required.

We may argue analogously for Q-homomorphisms from I into minimal right ideals contained in $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Q$ using conditions 3.3.2(iii) and (iv). We leave the reader to verify the necessity of the conditions.

If A is a proper right ideal of R, we define $l_N(A) = \{n \in N : nA = 0\}$, a left T-submodule of N. Clearly $l_Q\left(\begin{bmatrix} A & M \\ N & T \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ l_N(A) & 0 \end{bmatrix}$, so to ensure that Q is an S-ring we need to assume $l_N(A) \neq 0$. If A is a proper left ideal of R, we define $r_M(A) = \{m \in M : Am = 0\}$. Using similar notation for annihilators of proper right or left ideals of T in N or M respectively, to ensure Q is an S-ring, we clearly need

3.3.4 <u>Assumption</u>: If A is a maximal right (respectively left) ideal of R, then $l_N(A) \neq 0$ (respectively $r_M(A) \neq 0$), and if

B is a maximal left (respectively right) ideal of T, then $r_N(B) \neq 0$ (respectively $l_M(B) \neq 0$).

It is now clear that Q is an S-ring, so by 2.2.17 and 3.3.3, Q is a D-ring if and only if conditions 3.3.2(i) $\rightarrow$ (iv) and their left equivalents are satisfied. Before looking at some specific examples, we consider when Q is self-injective.

3.3.5 Definition: Let A be a right ideal of R and let $f:A \rightarrow R$ be an R-homomorphism. We say f is M-balanced if for any $a_1,\ldots,a_k \in A$, $m_1,\ldots,m_k \in M$ with $a_1m_1 + \ldots + a_km_k = 0$, we have $f(a_1)m_1 + \ldots + f(a_k)m_k = 0$. We similarly define an N-balanced T-homomorphism from a right ideal of T to T.

3.3.6 LEMMA:

Q is right self-injective if and only if for any submodules R_1, M_1, N_1, T_1 of R_R, M_T, N_R, T_T respectively, the following conditions are satisfied:

(i) Every T-homomorphism $f:M_1 \rightarrow M$ is given by left multiplication by an element of R.

(ii) Every M-balanced R-homomorphism $f:R_1 \rightarrow R$ is given by left multiplication by an element of R.

(iii) Every T-homomorphism $f:T_1 \rightarrow M$ is given by left multiplication by an element of M.

(iv), (v), (vi):- analogous conditions on R-homomorphisms $f:N_1 \rightarrow N$, $g:R_1 \rightarrow N$ and N-balanced T-homomorphisms $f:T_1 \rightarrow T$.

Proof: Suppose the conditions (i) $\rightarrow$ (vi) are satisfied, I is a right ideal of Q, and $h:I \rightarrow Q$ is a Q-homomorphism. In the same way as 3.3.3, we may assume $h(I) \leq \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix}$ and use conditions (i), (ii) and (iii). Let $I + \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & M \\ Y & B \end{bmatrix}$,

$I \cap \begin{bmatrix} 0 & 0 \\ N & T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Y_o & B_o \end{bmatrix}$, $I + \begin{bmatrix} 0 & 0 \\ N & T \end{bmatrix} = \begin{bmatrix} A & X \\ N & T \end{bmatrix}$,

$I \cap \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_o & X_o \\ 0 & 0 \end{bmatrix}$. Now $\begin{bmatrix} 0 & 0 \\ Y_o & 0 \end{bmatrix} \subseteq R \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so

$h\left(\begin{bmatrix} 0 & 0 \\ Y_o & 0 \end{bmatrix}\right) \subseteq \begin{bmatrix} R & M \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$, which contains

no non-zero right ideal of Q. Hence $\begin{bmatrix} 0 & 0 \\ Y_o & 0 \end{bmatrix} \subseteq$ Ker h. Clearly

now we can define an R-homomorphism $f: A \to R$ by $f(a) = r$

whenever $\begin{bmatrix} a & 0 \\ y & 0 \end{bmatrix} \in I$ and $h\left(\begin{bmatrix} a & 0 \\ y & 0 \end{bmatrix}\right) \in \begin{bmatrix} r & M \\ 0 & 0 \end{bmatrix}$. If

$a_1, \ldots, a_k \in R$, $m_1, \ldots, m_k \in M$, $y_1, \ldots, y_k \in Y$ with $a_1m_1 + \ldots + a_km_k = 0$,

$\begin{bmatrix} a_i & 0 \\ y_i & 0 \end{bmatrix} \in I$ for $1 \leqslant i \leqslant k$, then

$\begin{bmatrix} a_1 & 0 \\ y_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix} + \ldots + \begin{bmatrix} a_k & 0 \\ y_k & 0 \end{bmatrix}\begin{bmatrix} 0 & m_k \\ 0 & 0 \end{bmatrix} = 0$, so

$$h\left(\begin{bmatrix} a_1 & 0 \\ y_1 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix} + \ldots + h\left(\begin{bmatrix} a_k & 0 \\ y_k & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & m_k \\ 0 & 0 \end{bmatrix} = 0,$$

i.e. $f(a_1)m_1 + \ldots + f(a_k)m_k = 0$. Hence f is M-balanced. So by (ii), f is given by left multiplication by an element $r \in R$.

Let $h': I \to Q$ be the Q-homomorphism defined by

$$h'\left(\begin{bmatrix} a & x \\ y & b \end{bmatrix}\right) = h\left(\begin{bmatrix} a & x \\ y & b \end{bmatrix}\right) - \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & x \\ y & b \end{bmatrix} \in \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}.$$

Clearly h is given by left multiplication if and only if h' is, so we may assume $h(I) \subseteq \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$. But now

$I \cap \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \subseteq$ Ker h, and in a similar way to 3.3.3, using

(i) and (iii), we can deduce that h is given by left multiplication. It follows from 1.3.11 that Q is right self-injective. We leave the reader to verify the necessity of the conditions.

The results we have established enable us to give some specific examples of D-rings. We start with a non-self-injective example.

3.3.7 Example: Let $R = T = \mathbb{Z}_{(p)}$, and let M be as described in 3.2.6, and let N = M. Clearly R, T, M, N fit the general construction described, including assumptions 3.3.1 and 3.3.4 . Recall that R x M (see 3.1.2) is a D-ring (by 3.2.6), and that if $m \in M$, then every submodule of mR is finitely generated. It follows from 2.2.17 that conditions 3.3.2(i) $\rightarrow$ (iv) are satisfied, so Q is a D-ring. However, as observed in 3.2.6, condition 3.3.6(i) is not satisfied, so Q is not self-injective.

3.3.8 Example: Let S be a self-injective D-ring, and A an ideal of S such that $l_S(A) = r_S(A)$. Let $R = T = \frac{S}{A}$ and $M = N = l_S(A)$. Clearly R, T, M, N fit the general construction described, including assumptions 3.3.1 and 3.3.4 . Further, since S is self-injective, it is straightforward to verify that conditions 3.3.2(i) $\rightarrow$ (iv), 3.3.6(i) $\rightarrow$ (vi), and their left symmetries are satisfied. Hence Q is a self-injective D-ring.

3.3.9 Example: Let S be a self-injective D-ring and let A be an ideal of R. Let $R = \frac{S}{A}$ and $M = l_S(A)$, $N = r_S(A)$. Let T be the 'mirror image' of R (i.e. T = R as an abelian group, but if $r_1, r_2, r_3 \in R$ with $r_1 r_2 = r_3$ in R, then $r_2 r_1 = r_3$ in T). Clearly R, T, M, N fit the general construction described, including assumptions 3.3.1 and 3.3.4 . Further, since S is self-injective, it is straightforward to verify that conditions 3.3.2(i) $\rightarrow$ (iv), 3.3.6(i) $\rightarrow$ (vi), and their left symmetries are satisfied. Hence Q is a self-injective D-ring.

Chapter 4.

RINGS WHOSE PROPER HOMOMORPHIC IMAGES ARE I.P.R.I.-RINGS.

Let R be a prime bounded Noetherian ring, each of whose proper homomorphic images is a self-injective (hence Quasi-Frobenius) ring. By 2.3.7 the proper homomorphic images of R are Artinian p.r.i.- and p.l.i.-rings, so by a theorem of Hajarnavis (1.6.34), R is a Dedekind prime ring. In this chapter we will generalize this result by considering Noetherian rings whose proper homomorphic images are i.p.r.i.-rings (defined in 4.1.1).

The results of this chapter are the joint work of Dr. Hajarnavis and the author. In particular, following some results of the author, Dr. Hajarnavis observed the result given in theorem 4.2.8, and guided the author's further research.

We start this chapter by establishing some of Robson's results on i.p.r.i.-rings (see [32]).

§ 1 I.p.r.i.-rings.

4.1.1 Definition: A ring R is said to be an i.p.r.i.-ring (respectively i.p.l.i.-ring) if every ideal of R is principal as a right (respectively left) ideal.

The proof of the following lemma, known as Krull's Intersection Theorem, is identical to that given by Goldie for p.r.i.-rings in [12], lemma 3.1 . As observed by Robson in [32], Goldie's proof is equally valid for Noetherian i.p.r.i.-rings.

4.1.2 LEMMA ([32], corollary 3.2):

Let R be a Noetherian i.p.r.i.-ring, and let A be an ideal of R. Then there is an element $a \in A$ such that $(1 - a)(\bigcap_{n=1}^{\infty} A^n) = 0$.

Proof: $A = xR$ for some $x \in R$, so clearly $A^n = x^nR$ for each $n \in \mathbb{N}$. $\bigcap_{n=1}^{\infty} A^n$ is an ideal of R, so $\bigcap_{n=1}^{\infty} A^n = yR$ for some $y \in R$.

$y \in A^n$ for each $n \in \mathbb{N}$, so there are elements $r_1, r_2, \ldots \in R$ with $y = xr_1 = x^2r_2 = \ldots$. Since R is Noetherian, $r_{k+1} \in r_1R + .. + r_kR$ for some $k \in \mathbb{N}$, say $r_{k+1} = r_1s_1 + .. + r_ks_k$. Then
$$y = x^{k+1}r_{k+1} = x^{k+1}r_1s_1 + .. + x^{k+1}r_ks_k$$
$$= x^kys_1 + .. + xys_k \in xyR.$$

Hence $yR = xyR$. Clearly now $y = xb_1 = x^2b_2 = \ldots$ for some $b_1, b_2, \ldots \in yR$. Now R is Noetherian, so $r(x^t) = r(x^{t+1})$ for some $t \in \mathbb{N}$, and clearly $x^tR \cap r(x) = 0$. So $yR \cap r(x) = 0$. Thus $b_i = xb_{i+1}$ for each $i \in \mathbb{N}$, so $Rb_1 \subseteq Rb_2 \subseteq \ldots$. So $Rb_n = Rb_{n+1}$ for some $n \in \mathbb{N}$, $b_{n+1} = zb_n$ say. But $b_n = xb_{n+1}$, so $(1 - zx)b_{n+1} = 0$. Now $x^{n+1}y \in yR$, $x^{n+1}y = yr$ say, and $y = x^{n+1}b_{n+1}$, so $y - b_{n+1}r \in r(x^{n+1}) \cap yR = 0$. Thus $y = b_{n+1}r$ so $(1 - zx)y = 0$. Since $x \in A$, $zx \in A$ and since $yR = \bigcap_{n=1}^{\infty} A^n$, the result follows.

Lemma 4.1.2 immediately gives us:-

4.1.3 COROLLARY:

Let A be a proper ideal of a prime Noetherian i.p.r.i.-ring. Then $\bigcap_{n=1}^{\infty} A^n = 0$.

4.1.4 COROLLARY:

Let P be a prime ideal of a Noetherian i.p.r.i.-ring R. Then either P is a maximal ideal of R or an ideal direct summand of R.

Proof: Suppose $P + l(P) \neq R$, so $P + l(P) \subseteq M$ for some maximal ideal M of R. Suppose further $P \neq M$. Then $M = mR$ for some $m \in M$ and by 1.6.21 $m \in C(P)$. But $P \subseteq M = mR$, so $P = mP = m^2P = \ldots \subseteq \bigcap_{n=1}^{\infty} M^n$. Now by 4.1.2, $(1 - x)P = 0$ for some $x \in M$, so $1 \in x + l(P) \subseteq M$, a contradiction. Hence either $P = M$, a maximal ideal of R, or $P + l(P) = R$. Suppose $P + l(P) = R$, so $1 = e + y$ for some $e \in P$, $y \in l(P)$. For each $p \in P$, $yp \in l(P)P = 0$, so $p = (e + y)p = ep$, and in particular, $e = e^2$. Thus $P = eR$, so $l(P) = R(1 - e)$. Now $(1 - e)^2 = 1 - e$, so $\bigcap_{n=1}^{\infty} (l(P))^n = l(P)$, and by 4.1.2 there is an element $f \in l(P)$ with $(1 - f)l(P) = 0$. Thus for each $z \in l(P)$, $(1 - f)z = 0$, i.e. $z = fz$, and in particular, $f = f^2$. Hence $R(1 - e) = l(P) = fR$, whence $f = f(1 - e) = 1 - e$, so $l(P) = (1 - e)R$. Then $R = eR \oplus (1 - e)R = P \oplus l(P)$ as required.

4.1.5 Definition ([32], page 127): A ring R is said to be W-simple if $\frac{R}{W}$ is a simple ring, i.e. if W is a maximal ideal of R. Clearly in this case, $J = W$ is the unique maximal ideal of R, and an Artinian W-simple ring is a primary Artinian ring.

The following theorem was first proved by Robson in [32], theorem 3.6 . The proof given below combines 4.1.4 with the techniques of [32], theorem 2.2 .

4.1.6 THEOREM:

A Noetherian i.p.r.i.-ring is a (finite) direct sum of prime Noetherian i.p.r.i.-rings and W-simple Noetherian i.p.r.i.-rings.

Proof: Let R be an indecomposable Noetherian i.p.r.i.-ring. Clearly it suffices to show R is prime or W-simple. By 1.6.5 R has a finite number of minimal prime ideals, $P_1,..,P_n$ say, and $W = P_1 \cap .. \cap P_n$, no term in this expression being redundant. Suppose $n > 1$ and let $A = P_2 \cap .. \cap P_n$, so $P_1 \not\leq W$, $A \not\leq W$. Let $A = aR$, so by 1.6.21, $a \in C(P_1)$. Now $W = aR \cap P_1 = aP_1 = AP_1$, and $P_1A \leq A \cap P_1 = W = AP_1$, so $P_1A^k \leq A^kP_1$ for each $k \in \mathbb{N}$. Suppose inductively $k \in \mathbb{N}$ and $P_1{}^kA^k \leq W^k$. Then $P_1{}^{k+1}A^{k+1} = P_1{}^kP_1A^{k+1} \leq P_1{}^kA^{k+1}P_1 \leq W^kW = W^{k+1}$. Since W is nilpotent, induction shows $P_1{}^mA^m = 0$ for some $m \in \mathbb{N}$. Now $P_1 = pR$ for some $p \in R$ and clearly $p^kR = P_1{}^k$ for each $k \in \mathbb{N}$. Since R is Noetherian, $r(p^t) = r(p^{2t})$ for some $t \in \mathbb{N}$, $t \geq m$, so clearly $p^tR \cap r(p^t) = 0$. But $t \geq m$, so $P_1{}^tA^t = p^tA^t = 0$, i.e. $A^t \leq r(p^t)$. Thus $P_1{}^t \cap A^t = 0$. Since we are assuming R is indecomposable, 4.1.4 shows that $P_1,..,P_n$ are maximal ideals, so by 1.6.13 $P_1{}^t \oplus A^t = R$, and since $P_1 \not\leq W$, $A \not\leq W$, this is a contradiction. Hence $n = 1$, i.e. W is a prime ideal. Hence by 4.1.4 either $W = 0$, i.e. R is prime, or W is a maximal ideal, i.e. R is W-simple.

We will show that a Noetherian i.p.r.i.-ring has a quotient ring which is an Artinian p.r.i.-ring. Since, by 1.6.20, a prime Noetherian ring has a simple Artinian quotient ring, theorem 4.1.6 shows that it suffices to consider Noetherian W-simple i.p.r.i.-rings.

4.1.7 PROPOSITION ([32], lemmas 4.1 and 4.2):

Let R be a Noetherian W-simple ring. Then $C(0) = C(W)$, and R has a primary Artinian quotient ring Q. Further, $W(Q) = WQ = QW$.

Proof: Inductively suppose $k \in \mathbb{N}$ and $C(W) \subseteq C(W^k)$. Let $A = \{x \in R : (W + Rd)x \subseteq W^{k+1}$ for some $d \in C(W)\}$, a right ideal of R. Suppose $d \in C(W)$, $x \in A$, $(W + Rd)x \subseteq W^{k+1}$ and $r \in R$. By 1.6.20, $\frac{R}{W}$ has a quotient ring, so by 1.6.19 there are elements $d_1 \in C(W)$, $r_1 \in R$ with $d_1r - r_1d \in W$. Now $dx \in W^{k+1}$, and $d \in C(W) \subseteq C(W^k)$, so $x \in W^k$, whence $d_1rx \in r_1dx + Wx \subseteq W^{k+1}$. Hence A is an ideal of R. Now $A = a_1R + .. + a_tR$ for some $a_1,..,a_t \in R$, and there are elements $c_1,..,c_t \in C(W)$ with $(W + Rc_i)a_i \subseteq W^{k+1}$ for $1 \leq i \leq t$. Applying 1.6.21 to $\frac{R}{W}$ shows that $\bigcap_{i=1}^{t} (W + Rc_i) \supseteq W + Rc$ for some $c \in C(W)$ (since a finite intersection of essential left ideals is clearly an essential left ideal), so $(W + Rc)A \subseteq W^{k+1}$. Now $c \in C(W)$, so $c \notin W$, and since R is W-simple, $(W + Rc)R = R$. Thus $A = (W + Rc)RA = (W + Rc)A \subseteq W^{k+1}$. Now if $d \in C(W)$, $dx \in W^{k+1}$, then $d \in C(W^k)$ so $x \in W^k$, whence $(W + Rd)x \subseteq W^{k+1}$, so $x \in A \subseteq W^{k+1}$. This and symmetry gives $C(W) \subseteq C(W^{k+1})$. Since W is nilpotent, $C(W) \subseteq C(0)$ by induction. Now for some $n \in \mathbb{N}$, $W^n = 0$ and $W^{n-1} \neq 0$. Clearly $l(W^{n-1})$ is a proper ideal of R containing W, so $l(W^{n-1}) = W$. Suppose $c \in C(0)$, $x \in R$, $cx \in W$. Then $cxW^{n-1} \subseteq W^n = 0$, so $xW^{n-1} = 0$, i.e. $x \in l(W^{n-1}) = W$. This and symmetry gives $C(0) \subseteq C(W)$. Thus $C(0) = C(W)$, and by 1.6.22, R has an Artinian quotient ring Q. Clearly $W(Q) \cap R \subseteq W$, and by 1.6.25, $W(Q) \subseteq WQ \cap QW$, and QW and WQ are ideals of Q. Thus $WQ = QWQ = QW$, whence $(WQ)^2 = W^2Q$, and inductively $0 = W^nQ = (WQ)^n$ for each $n \in \mathbb{N}$. Thus $WQ = W(Q) = QW$. If B is an ideal of Q, then clearly $B \cap R$ is an ideal of R, and since R is W-simple, it follows that WQ is a maximal ideal of Q. Hence Q is a primary Artinian ring, as required.

4.1.8 <u>THEOREM</u> ([32], theorem 2.4):

Let R be a primary Artinian ring, and suppose $W = zR$ for some $z \in R$. Then there is a completely primary Artinian p.r.i.-ring T and $n \in \mathbb{N}$ with $R \cong T_n$, the nxn matrix ring over T. Further, the only ideals of R are the powers of W.

<u>Proof</u>: Let $1 = e_1 + .. + e_n$, a sum of mutually orthogonal primitive idempotents of R. Now R is W-simple, so $R = W + Re_1R$, whence $e_1Re_i \not\subseteq W$ for each i. By 1.5.1, there are elements $u_{1i} \in e_1Re_i$, $v_{i1} \in e_iRe_1$ with $u_{1i}v_{i1} = e_1$, $v_{i1}u_{1i} = e_i$ for $1 \leq i \leq n$. Let $T = e_1Re_1$ and define $f:R \rightarrow T_n$ by $f(x) = (u_{1i}xv_{j1}) \in T_n$ for all $x \in R$. It is straightforward to check that f is a ring isomorphism, and clearly T is a completely primary Artinian ring. Now $W = J$, so (by 1.2.2) $\frac{W}{W^2}$ is a completely reducible right R-module. Suppose $M_1,..,M_t$ are right ideals of R strictly containing W^2 with $\frac{W}{W^2} = \frac{M_1}{W^2} \oplus .. \oplus \frac{M_t}{W^2}$. Define $A = \{a \in R:za \in W^2\}$ and

$A_i = \{a \in R:za \in M_i\}$ for each i. Now $W^2 \subset M_i \subseteq W = zR$, so clearly $W \subseteq A \subset A_i$ and $M_i = zA_i$ for each i. Hence $A = fR + W$ for some $f = f^2 \in R$, and there are idempotents $f_1,..,f_t \in R$ with $ff_i = f_if = 0$ and $A_i = (f + f_i)R + W$ for each i. Suppose $r_1,..,r_t \in R$ and $f_1r_1 + .. + f_tr_t \in W = zR$. Then $zf_1r_1 + .. + zf_tr_t \in W^2$, and each $zf_ir_i \in zA_i = M_i$. But $\frac{M_1}{W^2} + .. + \frac{M_t}{W^2}$ is a direct sum, so $zf_ir_i \in W^2$ for each i. So

$f_i r_i \in A = fR + W$, and since $f_i f = 0$, $f_i r_i \in W$ for each i.

Hence $\frac{f_1R + W}{W} + \dots + \frac{f_tR + W}{W}$ is a direct sum in $\frac{R}{W}$. Thus $t \leq n$, so $\dim\left(\left(\frac{W}{W^2}\right)_R\right) \leq n$. But $R \cong T_n$, so writing $X = W(T) = e_1We_1$, $\dim\left(\left(\frac{W}{W^2}\right)_R\right) = n.\dim\left(\left(\frac{X}{X^2}\right)_T\right)$. Clearly now $\dim\left(\left(\frac{X}{X^2}\right)_T\right) = 1$, i.e. $\frac{X}{X^2}$ is a simple right T-module. Choose $x \in X$, $x \notin X^2$ (or if $X = 0$, put $x = 0$). Now $X = xT + X^2$, so $\left(\frac{X}{xT}\right)X = \frac{X}{xT}$. But $X = W(T) = J(T)$, so by Nakayama's Lemma (1.1.4), $\frac{X}{xT} = 0$, i.e. $X = xT$. Suppose I is a proper non-zero right ideal of T, so for some $k \in \mathbb{N}$, $I \subseteq x^kT$, $I \not\subseteq x^{k+1}T$. Define $K = \{a \in T : x^ka \in I\}$, a right ideal of T, and $K \not\subseteq xT$. But T is a completely primary Artinian ring, so $K = T$. So $I = x^kK = x^kT$. Hence T is a p.r.i.-ring, the only (right) ideals being the powers of $xT = J(T)$. The final statement of the theorem follows immediately from the isomorphism $R \cong T_n$.

It is well known that if T is a p.r.i.-ring and $n \in \mathbb{N}$, then the nxn matrix ring T_n is also a p.r.i.-ring (see [20], page 77, or [32], theorem 8.5). Hence we have the following corollary to 4.1.8 .

4.1.9 <u>COROLLARY</u>:

Let R be a primary Artinian ring, and suppose $W = zR$ for some $z \in R$. Then R is a p.r.i.-ring.

As shown by Robson in [32], theorem 4.5, we can now deduce:-

4.1.10 <u>COROLLARY</u>:

A Noetherian i.p.r.i.-ring has a quotient ring which is an Artinian p.r.i.-ring.

<u>Proof</u>: Combine 1.6.20, 4.1.6, 4.1.7, and 4.1.9 .

4.1.11 <u>COROLLARY</u>:

Let R be a Noetherian W-simple ring, and suppose $W = zR$ for some $z \in R$. Then the only ideals of R are the powers of W, and R is an i.p.r.i.-ring.

<u>Proof</u>: By 4.1.7, R has a primary Artinian quotient ring Q, and $W(Q) = WQ = zQ$, so by 4.1.8, the only ideals of Q are those of the form $(W(Q))^k = W^kQ$ for some $k \in \{0\} \cup \mathbb{N}$. Suppose A is a proper ideal of R. Applying 4.1.7 to $\frac{R}{A}$ and R, we see that $C(A) = C(W) = C(0)$, so by 1.6.25, AQ is an ideal of Q and $A = AQ \cap R$. Thus there is an integer $k \in \mathbb{N}$ with $AQ = W^kQ$. So $A = AQ \cap R = W^kQ \cap R$. But for the same reason, $W^k = W^kQ \cap R$. Therefore, $A = W^k = z^kR$ as required.

Our next theorem was first proved by Robson in [32], section 5, under symmetric conditions.

4.1.12 <u>THEOREM</u>:

Let R be a Noetherian i.p.r.i.-ring. Then

(i) The ideals of R commute.

(ii) R has primary decomposition.

(iii) An ideal of R is primary if and only if it is the power of a prime ideal.

(iv) Every ideal of R can be expressed as a product of prime ideals of R.

<u>Proof</u>: If T_1, T_2 are rings satisfying (i) $\rightarrow$ (iv), then clearly so is $T_1 \oplus T_2$. By 4.1.11, it is clear that a Noetherian W-simple i.p.r.i.-ring satisfies (i) $\rightarrow$ (iv), so by 4.1.6, we may assume R is prime. It follows from 4.1.4 that the only non-zero prime ideals of R are the maximal ideals of R, so by 4.1.6 every proper factor ring of R is a direct sum of Noetherian W-simple i.p.r.i.-rings, so every proper factor ring of R satisfies (i) $\rightarrow$ (iv). Suppose A, B are non-zero ideals of R. Now R is prime, so $AB \cap BA \neq 0$. Applying (i) to $\frac{R}{AB \cap BA}$ gives $AB + (AB \cap BA) = BA + (AB \cap BA)$, i.e. $AB = BA$. So (i) holds in R. If A is a non-zero meet-irreducible ideal of R, then applying (ii) to $\frac{R}{A}$ shows A is primary. Since 0 is a prime, hence primary, ideal of R, clearly (ii) holds in R. Applying (iv) to $\frac{R}{A}$, we see that there are non-zero prime ideals $P_1, .., P_n$ of R, and $k_1, .., k_n \in \mathbb{N}$ with $A \supseteq P_1^{k_1} P_2^{k_2} \ldots P_n^{k_n} = B$ say. Clearly we may choose $k_1 + \ldots + k_n$ minimal, and, since $B \neq 0$, applying (iv) again to $\frac{R}{B}$, we clearly get $A = P_1^{k_1} P_2^{k_2} \ldots P_n^{k_n}$, so (iv) holds in R. (iii) follows immediately from (iv) and the definition of a primary ideal.

§ 2 Noetherian rings whose proper homomorphic images are i.p.r.i.-rings.

4.2.1 PROPOSITION:

Let R be a prime Noetherian ring whose proper homomorphic images are i.p.r.i.-rings. Then

(i) The ideals of R commute.

(ii) R has primary decomposition.

(iii) An ideal of R is primary if and only if it is the power of a prime ideal.

(iv) Every ideal of R can be expressed as a product of prime ideals of R.

Proof: By 4.1.12, (i) $\longrightarrow$ (iv) hold in every proper factor ring of R, so in the same way as in 4.1.12, we can verify (i) $\longrightarrow$ (iv) for R.

4.2.2 LEMMA:

Let R be a Noetherian ring whose proper homomorphic images are i.p.r.i.-rings, and let P be a non-zero prime ideal of R. Then

either (i) P is a maximal ideal of R, $P \neq P^2$

or (ii) P is a minimal non-zero prime ideal, and $P = P^2$

or (iii) P is a minimal ideal, and $P^2 = 0$.

Proof: Suppose A is a non-zero ideal of R with $P^2 \subseteq A < P$ (if such exists). Applying 4.1.4 to $\frac{R}{A}$ shows that P is a maximal ideal of R, so (i) holds. Suppose no such ideal A exists, so either (iii) holds or $P = P^2$. In the latter case, if Q is a prime ideal of R, $0 \neq Q \subseteq P$, then by 4.1.3 applied to $\frac{R}{Q}$, $\bigcap_{n=1}^{\infty} P^n \subseteq Q$. But $P = P^2 = \dots = \bigcap_{n=1}^{\infty} P^n$, so $P = Q$. Hence if $P = P^2$, then P is a minimal non-zero prime ideal, so (ii) holds.

4.2.3 LEMMA:

Let R be a Noetherian ring whose proper homomorphic images are i.p.r.i.-rings, and let P, Q be prime ideals of R. Then either $P \subseteq Q$, $Q \subseteq P$ or $Q + P = R$.

Proof: Suppose $P \nsubseteq Q$ and $Q \nsubseteq P$, and suppose M is a maximal ideal of R with $P + Q \subseteq M$. Since $P \nsubseteq Q$, $Q \nsubseteq P$ clearly neither P nor Q are maximal ideals of R. If $P^2 = 0 \subseteq Q$, then $P \subseteq Q$ since Q is prime, so $P^2 \neq 0$ and similarly $Q^2 \neq 0$. By 4.2.2, we must have $P = P^2$ and $Q = Q^2$. But now $P = \bigcap_{n=1}^{\infty} P^n \subseteq \bigcap_{n=1}^{\infty} M^n$, and $Q \subseteq \bigcap_{n=1}^{\infty} M^n$, and applying 4.1.3 to the prime factor rings $\frac{R}{P}$ and $\frac{R}{Q}$ we see that $\bigcap_{n=1}^{\infty} M^n \subseteq P$ and $\bigcap_{n=1}^{\infty} M^n \subseteq Q$. Hence $P = Q$, a contradiction, which completes the proof.

So far we have investigated the ideal structure of rings whose proper homomorphic images are i.p.r.i.-rings. In order to progress further in the prime case, we need information on the one-sided ideal structure of such rings. To this end, we introduce an extra hypothesis, namely boundedness. Our next lemma, which is similar to [14], lemma 3.1, is a technical result which helps us to use the assumption of boundedness effectively.

4.2.4 LEMMA:

Let R be a prime bounded Noetherian ring such that

(i) the prime ideals of R commute, and

(ii) if P, Q are prime ideals of R, then either $P \subseteq Q$, $Q \subseteq P$, or $P + Q = R$.

Let I be a meet-irreducible essential right ideal of R, and define $P_I = \{x \in R: Ax \subseteq I \text{ for some right ideal } A \not\subseteq I\}$. Then P_I is an ideal of R, and there is a non-zero prime ideal Q of R and $n \in \mathbb{N}$ such that $Q \subseteq P_I$ and $Q^n \subseteq I$.

<u>Proof</u>: Suppose $x, y \in P_I$, and A, B are right ideals of R, $A \not\subseteq I$, $B \not\subseteq I$, with $Ax \subseteq I$, $By \subseteq I$. Since I is meet-irreducible, $I \subset (A + I) \cap (B + I)$, and clearly $[(A + I) \cap (B + I)](x + y) \subseteq I$. So $x + y \in P_I$. Clearly now P_I is an ideal of R. Now R is bounded and I is essential, so I contains a non-zero ideal, and by 1.6.6, I contains a product of non-zero prime ideals, $0 \neq Q_1Q_2..Q_n \subseteq I$ say. Clearly we may assume n is minimal, i.e. I does not contain a product of $n - 1$ non-zero prime ideals. Now for each i, $Q_1..Q_{i-1}Q_{i+1}..Q_n \not\subseteq I$ and (using (i)) $(Q_1..Q_{i-1}Q_{i+1}..Q_n)Q_i \subseteq I$, so $Q_i \subseteq P_I$. Thus $Q_1 + .. + Q_n \subseteq P_I$. Now $1 \notin P_I$, so (by (ii)) for any i, j either $Q_i \subseteq Q_j$ or $Q_j \subseteq Q_i$. But now we may put $Q = \min\{Q_1,..,Q_n\}$, a non-zero prime ideal of R, and $Q \subseteq P_I$, $Q^n \subseteq I$ as required.

Since we also need the left-handed version of lemma 4.2.4, we restricted the stated hypotheses (i) and (ii) to symmetric properties of a ring. However, if R is a prime bounded Noetherian ring whose proper homomorphic images are i.p.r.i.-rings, then 4.2.1 and 4.2.3 show that R satisfies hypotheses (i) and (ii) of 4.2.4 .

4.2.5 PROPOSITION:

Let R be a prime bounded Noetherian ring whose proper homomorphic images are i.p.r.i.-rings, and let P be a non-zero prime ideal of R. Then

(i) P is a maximal ideal of R.

(ii) $\bigcap_{n=1}^{\infty} P^n = 0$.

(iii) For all $n \in \mathbb{N}$, $C(P) = C(P^n) \subseteq C(0)$.

Proof: Suppose A is a non-zero proper ideal of R and $A = A^2$. R is prime, so by 1.6.21 we may choose $c \in C(0) \cap A$. By 1.6.10, $cA = I_1 \cap .. \cap I_k$ for some meet-irreducible right ideals $I_1, .., I_k$ of R. Clearly $c^2 \in C(0) \cap cA \subseteq I_i$, so by 1.6.21 each I_i is essential. Since $A \neq R$ and $c \in C(0)$, $cA \neq cR$, so for some j, $1 \leq j \leq k$, $cR \nsubseteq I_j$. Define P_{I_j} as in 4.2.4, so there is a non-zero prime ideal Q of R and $n \in \mathbb{N}$ with $Q \subseteq P_{I_j}$ and $Q^n \subseteq I_j$. Now $cR \nsubseteq I_j$, $cA = cRA \subseteq I_j$ so $A \subseteq P_{I_j}$, whence $A + Q \subseteq P_{I_j}$. If $A \subseteq Q$ then $c \in A = A^2 = .. = A^n \subseteq Q^n \subseteq I_j$, so $cR \subseteq A \subseteq I_j$, a contradiction. Thus $A \nsubseteq Q$. Suppose first A is prime, so by 4.2.2 A is a minimal non-zero prime ideal. But $A \nsubseteq Q$, so by 4.2.3, $R = A + Q \subseteq P_{I_j}$, a contradiction. Hence if P is a non-zero prime ideal then $P \neq P^2$, so by 4.2.2 P is a maximal ideal. But now, even if A is not prime, $A + Q \subseteq P_{I_j}$, $A \nsubseteq Q$, and Q is a non-zero prime, hence a maximal ideal of R. Hence $R = A + Q \subseteq P_{I_j}$, a contradiction. Thus if A is any non-zero proper ideal of R then $A \neq A^2$. Let P be a non-zero prime ideal of R and let $A = \bigcap_{n=1}^{\infty} P^n$, and suppose

$A \neq 0$. Then $A^2 \neq 0$. Now $\frac{A}{A^2} = \bigcap_{n=1}^{\infty} \frac{P^n}{A^2}$. But by 4.1.6, $\frac{R}{A^2}$ is a direct sum of Noetherian W-simple i.p.r.i.-rings, so $\left(\frac{A}{A^2}\right)^2 = \frac{A}{A^2}$. Hence $A = A^2$, a contradiction. So $\bigcap_{n=1}^{\infty} P^n = 0$.

Applying 4.1.7 to $\frac{R}{P^n}$ for each $n \in \mathbb{N}$ clearly gives $C(P) = C(P^n)$. Suppose $c \in C(P)$, $x \in R$ and $cxc = 0$. Now for all $n \in \mathbb{N}$, $c \in C(P^n)$ and $cxc \in P^n$, so $x \in P^n$. So $x \in \bigcap_{n=1}^{\infty} P^n = 0$.

Thus $C(P) \subseteq C(0)$ as required.

The proof of our next result is a combination of [14], proposition 3.2, and the proof of [13], theorem 2.58.

4.2.6 PROPOSITION ([14], lemma 3.3):

Let R be a prime bounded Noetherian ring such that

(i) the non-zero prime ideals of R are maximal ideals, and

(ii) the maximal ideals of R commute.

Let P be a non-zero prime ideal of R. Then R satisfies the Ore condition with respect to C(P).

Proof: Suppose $a \in R$, $c \in C(P)$. For each $n \in \mathbb{N}$, $\frac{R}{P^n}$ is a W-simple ring, so by 4.1.7 and 1.6.19, there are elements $a_n \in R$, $c_n \in C(P)$, and $p_n \in P^n$ with $ac_n - ca_n = p_n$. Let $I = p_1R + p_2R + \dots$, and let K be a right ideal of R maximal with respect to $I \cap K = 0$, so clearly $I \oplus K$ is an essential right ideal of R. It follows from 1.6.21 that $IP \oplus K \supseteq (I \oplus K)P$ is an essential right ideal, so by 1.6.10, $IP \oplus K = I_1 \cap \dots \cap I_t$ for some meet-irreducible essential right ideals $I_1, \dots, I_t$ of R. Suppose $1 \leq j \leq t$. Define P_{I_j} as in 4.2.4. Since non-zero prime ideals are maximal, 4.2.4 shows that for some $k_j \in \mathbb{N}$,

$P_{I_j}^{k_j} \leq I_j$. If $I \leq I_j$ clearly $I \cap P^{k_j} \leq I_j$. If $I \neq I_j$, then since $IP \leq I_j$, $P \leq P_{I_j}$, so $P^{k_j} \leq I_j$. Hence in either case, $I \cap P^{k_j} \leq I_j$. Putting $k = \max\{k_1, \ldots, k_t\}$, we see that $I \cap P^k \leq I_1 \cap \ldots \cap I_t = IP \oplus K$. But $(IP \oplus K) \cap I = IP$, so $I \cap P^k \leq IP$. Now R is Noetherian, so $I = p_1R + \ldots + p_mR$ for some $m \in \mathbb{N}$. Choose $n \geq \max\{k, m\}$. Now
$p_n \in I \cap P^n \leq IP = p_1P + \ldots + p_mP$, so $p_n = p_1q_1 + \ldots + p_mq_m$ for some $q_1, \ldots, q_m \in P$. Now for $1 \leq i \leq m$, $ac_iq_i = ca_iq_i + p_iq_i$, so $a(c_1q_1 + \ldots + c_mq_m) = c(a_1q_1 + \ldots + a_mq_m) + p_n$. But
$ac_n = ca_n + p_n$. Thus
$a(c_n - c_1q_1 - \ldots - c_mq_m) = c(a_n - a_1q_1 - \ldots - a_mq_m)$. But
$c_n \in C(P)$ and $q_1, \ldots, q_m \in P$, so $c_n - c_1q_1 - \ldots - c_mq_m \in C(P) + P$
$= C(P)$.

Hence R satisfies the right Ore condition with respect to $C(P)$. The left Ore condition follows by symmetry.

Let R be a prime bounded Noetherian ring whose proper homomorphic images are i.p.r.i.-rings, and let P be a non-zero prime ideal of R. By 4.2.1, 4.2.5, 4.2.6, and 1.6.24 we can localise R at P. Since R_P is both the right and the left partial quotient ring of R with respect to $C(P)$, 1.6.25 shows that R_P is both right and left Noetherian. Now
$J(R_P) = PR_P$ $(= R_PP)$, a principal right ideal of R_P. As observed by Hajarnavis, it follows from [31], theorem 3.6, that PR_P is also a principal left ideal of R, so by [15], proposition 1.3, R_P is a Dedekind prime ring. Hence by 1.6.30 and 1.6.31, R is a Dedekind prime ring. The results quoted are proved for a more general case than we need here, so, for completeness, we use our extra conditions to provide an easy

proof of these facts.

4.2.7 THEOREM:

Let R be a local prime bounded Noetherian ring, and suppose $J = xR$ for some $x \in R$ and $\bigcap_{n=1}^{\infty} J^n = 0$. Then $J = Rx$, the only non-zero ideals of R are the powers of J, and R is a Dedekind p.r.i.- and p.l.i.-ring.

Proof: If $J = 0$, R is simple Artinian and the result is trivial, so suppose $x \neq 0$. R is prime, so $x \in C(0)$, and since R has a quotient ring Q by 1.6.20, we may write x^{-1} $(\in Q)$ without confusion. Let A be a non-zero proper ideal of R. $0 \neq A$, so for some $k \in \mathbb{N}$, $A \leq x^kR$ and $A \not\leq x^{k+1}R$. Then $x^{-k}A \leq R$ and $x^{-k}A \not\leq xR$. Applying 4.1.11 to $\frac{R}{x^{k+1}R}$ gives $A + x^{k+1}R = x^kR$, so $x^{-k}A + xR = R$. But $xR = J$ is superfluous in R, so $x^{-k}A = R$. Thus $A = x^kR$. Hence every non-zero ideal of R is a power of J. Now (by 1.6.21) $x \in C(0)$, so Rx is an essential left ideal of R. But R is bounded, so for some $k \in \mathbb{N}$, $x^kR \leq Rx$. Clearly we may assume k is minimal with respect to this property. If $k \neq 1$, then $B = \{r \in R : rx \in x^kR\}$ is a proper ideal of R, since xR is an ideal of R, and $B \leq xR$ since R is local. But $x^kR \leq Rx$, so $x^kR = Bx$, and hence $x^kR \leq xRx$. Then $x^{k-1}R \leq Rx$, contradicting the choice of k. Thus $k = 1$ and $xR \leq Rx$. But $x \in J$, so $Rx \leq J = xR$. So $xR = Rx$. Suppose I is an essential right ideal of R. R is bounded, so I contains a non-zero ideal, J^k say. Now by 1.6.3, $\frac{R}{J^{k+1}}$ is Artinian, so by 4.1.9, $\frac{R}{J^{k+1}}$ is a p.r.i.-ring. Thus $I = aR + J^{k+1}$ for some $a \in R$. But $J^k \leq I$, so $(\frac{I}{aR})J = \frac{I}{aR}$, and by Nakayama's Lemma (1.1.4), $\frac{I}{aR} = 0$, i.e. $I = aR$. Since I is essential, $a \in C(0)$, so $I_R \cong R_R$, a projective right R-module. Since any right ideal of R is a direct summand of an essential right ideal of R, it follows that every right ideal of R is principal and projective. Since we have established $J = Rx$, the result now follows by symmetry.

4.2.8 THEOREM:

Let R be a prime bounded Noetherian ring whose proper homomorphic images are i.p.r.i.-rings. Then R is a Dedekind prime ring. If, further, R itself is an i.p.r.i.-ring, then R is also an i.p.l.i.-ring.

Proof: Let P be a non-zero prime ideal of R. By 4.2.1, 4.2.5, 4.2.6, and 1.6.24, we can localise R at P, obtaining a Noetherian local ring R_P. It is easy to verify that R_P is also a prime ring. Now $J(R_P) = PR_P = R_PP$. $\frac{R}{P^2}$ is an i.p.r.i.-ring, so $P = zR + P^2$ for some $z \in R$, whence $PR_P = zR_P + P^2R_P$. Thus $(\frac{PR_P}{zR_P})PR_P = \frac{PR_P}{zR_P}$, so by Nakayama's lemma (1.1.4), $PR_P = zR_P$. Suppose I is an essential right ideal of R_P. Then (by 1.6.21), $I \cap C_{R_P}(0) \neq \emptyset$, so clearly $I \cap C_R(0) \neq \emptyset$, whence $I \cap R$ is an essential right ideal of R. Now by 4.2.1, $I \cap R$ contains a product of commuting non-zero prime ideals of R, each maximal ideals by 4.2.5, so $I \cap R \supseteq P^kP_1..P_n$, for some $k \in \mathbb{N}$ and some maximal ideals $P_1,..,P_n$ distinct from P. Clearly $P_i \cap C_R(P) \neq \emptyset$ for $1 \leq i \leq n$, so $I \supseteq P^kP_1..P_nR_P$, a non-zero ideal. This and a symmetrical argument shows that R_P is bounded, so by 4.2.7, R_P is a Dedekind prime ring. It follows by 1.6.31 that R is an Asano order, and since R is bounded, 1.6.30 shows that R is a Dedekind prime ring. Finally, suppose also that R is an i.p.r.i.-ring and $A = aR$ is a non-zero ideal of R. Then $a \in C_R(0)$, so $a^{-1} \in A^{-1}$. But now $Aa^{-1} \leq AA^{-1} \leq R$, so $A \leq Ra$. Since $a \in A$, clearly $A = Ra$. Hence R is an i.p.l.i.-ring as required.

Theorem 4.2.8 completes our study of prime Noetherian rings whose proper homomorphic images are i.p.r.i.-rings, and we now finish this section with a partial structure theory for the non-prime case.

4.2.9 THEOREM:

Let R be a Noetherian ring whose proper homomorphic images are i.p.r.i.-rings. Then

either (i) R is an i.p.r.i.-ring

or (ii) W is a prime ideal of R, and

either (a) R is prime,

or (b) W is a minimal ideal of R,

or (c) R is W-simple, $W^2 = 0$, and if $W = A_0 \supset A_1 \supset \ldots \supset A_n = 0$ is a chain of ideals of maximal length, then $n = 2$.

or (iii) $R \simeq \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ for some prime Noetherian i.p.r.i.-rings S and T, and for some non-zero $S - T$ bimodule M satisfying

(a) M contains no non-zero proper sub-bimodule

(b) if $0 \neq s \in S$, $0 \neq t \in T$, then $sM \neq 0$ and $Mt \neq 0$.

Proof: Suppose A, B are non-zero ideals of R with $A \oplus B = R$. Then $A \cong \frac{R}{B}$ and $B \cong \frac{R}{A}$ are i.p.r.i.-rings, whence clearly

(i) holds. Suppose then R is indecomposable. Suppose first W is a prime ideal of R, so by 4.2.2 either (ii)(a) or (ii)(b) hold, or R is W-simple. Suppose then R is W-simple. First, consider the case $W^2 \neq 0$. Then $\frac{R}{W^2}$ is an i.p.r.i.-ring, so $W = zR + W^2$ for some $z \in R$. But now $(\frac{W}{zR})W = \frac{W}{zR}$, so by Nakayama's Lemma (1.1.4) $W = zR$, whence by 4.1.11, (i) above holds. Suppose now R is W-simple and $W^2 = 0$. Let A be a non-zero proper ideal of R, so clearly $A \subseteq W$. Now $W^2 = 0$ and $\frac{R}{A}$ is an i.p.r.i.-ring, so by 4.1.11 either $\frac{W}{A}$ is a minimal ideal of R or $A = W$. Clearly now this case is covered either by (ii)(b) or (ii)(c). We now consider the case when W is not prime. By 1.6.5, there are minimal prime ideals $P_1, \ldots, P_n$ of R such that $W = P_1 \cap \ldots \cap P_n$, and no P_i is redundant in this expression. Since we are assuming W is not prime, $n > 1$. Now $W^k = 0$ for some $k \in \mathbb{N}$. Let $B = P_2^k \cap \ldots \cap P_n^k \nsubseteq W$, and let $M = P_1^k \cap \ldots \cap P_n^k = P_1^k \cap B \subseteq W$. Suppose A is a non-zero ideal of R and $A \subseteq M$. Then clearly $\frac{P_1}{A}, \ldots, \frac{P_n}{A}$ are the minimal prime ideals of the i.p.r.i.-ring $\frac{R}{A}$, and since $W^k = 0$, it follows from 4.1.6 that $\frac{M}{A} = \frac{P_1^k}{A} \cap \ldots \cap \frac{P_n^k}{A} = 0$, i.e. that $M = A$. Hence M is a minimal ideal or $M = 0$. Now each P_i is a minimal prime ideal of R, and R is indecomposable, so it follows from 4.2.3 and 1.6.13 that $M \neq 0$ and $\frac{R}{M} = \frac{P_1^k}{M} \oplus \frac{B}{M}$. Hence M is a minimal ideal, and by 1.2.14 there is an idempotent $e \in R$ with $P_1^k = eR + M = Re + M$ and $B = (1 - e)R + M = R(1 - e) + M$. Suppose $P_1 \nsubseteq l(M)$ and $P_1 \nsubseteq r(M)$, so by the minimality of M $P_1 M = M = MP_1$. Clearly $M^2 = 0$, so $M = P_1 M = P_1^2 M = \ldots = P_1^k M = (eR + M)M = eM \subseteq eR$, and similarly

$M \subseteq Re$. But now $P_1^k = eR = Re$, so $R = P_1^k \oplus l(P_1^k)$, contradicting the assumption that R is indecomposable. Thus $P_1 \subseteq l(M)$ or $P_1 \subseteq r(M)$. Similarly, $P_i \subseteq l(M)$ or $P_i \subseteq r(M)$ for $i = 2,..,n$. But $l(M) \neq R$, $r(M) \neq R$, so by 4.2.3 $l(M)$ and $r(M)$ each contain at most one minimal prime ideal. Since $n > 1$, clearly now $n = 2$ and either $P_1 \subseteq r(M)$, $P_2 \subseteq l(M)$ or $P_2 \subseteq r(M)$, $P_1 \subseteq l(M)$. Without loss of generality, assume $P_1 \subseteq r(M)$ and $P_2 \subseteq l(M)$, so $P_1 \not\subseteq l(M)$ and $P_2 \not\subseteq r(M)$. Now by the minimality of M, $M = P_1M = .. = P_1^kM = (eR + M)M \subseteq eR$, so $P_1^k = eR$. Similarly, $P_2^k = B = R(1 - e)$. Then
$M = P_1^k \cap P_2^k = eR \cap R(1 - e) = eR(1 - e)$, and
$(1 - e)Re \subseteq P_1^k \cap P_2^k = M \subseteq eR$, so $(1 - e)Re = 0$. Hence $eRe = eRe + (1 - e)Re = Re$, and $(1 - e)R(1 - e) = (1 - e)R$. Clearly now $eRe \simeq \frac{R}{P_2^k}$ and $(1 - e)R(1 - e) \simeq \frac{R}{P_1^k}$, and $M = eR(1 - e)$ is an $eRe - (1 - e)R(1 - e)$ bimodule. Let $S = eRe$ and let $T = (1 - e)R(1 - e)$, so S and T are Noetherian i.p.r.i.-rings. Define $R' = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ (with the obvious ring operations). Clearly the map $x \mapsto \begin{bmatrix} exe & ex(1-e) \\ 0 & (1-e)x(1-e) \end{bmatrix}$ defines a ring isomorphism $R \to R'$, so R' satisfies the hypotheses of the theorem. Since M is a minimal ideal of R, clearly M is a non-zero $S - T$ bimodule which contains no non-zero proper sub-bimodule. Let $C = \{s \in S: sM = 0\}$ and let $D = \{t \in T: Mt = 0\}$. Clearly C and D are proper ideals of S and T respectively, and $X = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ is an ideal of R'.

Let $S' = \frac{S}{C}$ and $T' = \frac{T}{D}$. Suppose $X \neq 0$. Then $\begin{bmatrix} 0 & M \\ 0 & T' \end{bmatrix}$ is an ideal of $\begin{bmatrix} S' & M \\ 0 & T' \end{bmatrix} \simeq \frac{R'}{X}$, an i.p.r.i.-ring, so

$\begin{bmatrix} 0 & 0 \\ 0 & 1_{T'} \end{bmatrix} \in \begin{bmatrix} 0 & m \\ 0 & t \end{bmatrix} \begin{bmatrix} S' & M \\ 0 & T' \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & T' \end{bmatrix}$ for some $m \in M$, $t \in T$.

Hence there is an element $x \in T'$ with $mx = 0$, $tx = 1_{T'}$. Now by 1.5.2, xt is an idempotent of T' and $xtT' \cong txT' = T'$, and since T' is Noetherian, a dimension argument shows $xt = 1_{T'}$. Thus $m = mxt = 0$, so $\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} S' & M \\ 0 & T' \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & T' \end{bmatrix}$, a contradiction since $M \neq 0$. Therefore, $X = 0$, i.e. $C = 0$ and $D = 0$. Finally, suppose $W(S) \neq 0$. Then $C = 0$, so by the minimality of M, $M = W(S)M$. But $(W(S))^k = 0$, so $M = W(S)M = \ldots = (W(S))^k M = 0$, a contradiction. But $S = eRe \cong \frac{R}{P_2^{\ k}}$, so $P_2 = P_2^{\ k}$, whence S is a prime ring. Similarly, T is a prime ring, and since $R \cong R'$, we see that R is as described in (iii).

§ 3 Examples.

In this section we give examples of the classes of rings studied in this chapter. We start with examples of the prime case, and in particular, examples of Asano orders and Dedekind prime rings.

4.3.1 Example: Let F be a field of characteristic zero, and let x, y be indeterminates over F which commute with the elements of F. Let $R = F[x,y]$ be the ring of polynomials in x and y with coefficients in F, subject to the relation $xy - yx = 1$. Then (see [43]) R is known to be a simple hereditary Noetherian domain which is not a division ring. Thus xR is an essential proper right ideal, and since R is simple, xR contains no non-zero ideal. Hence R is not bounded. However, R is simple, so clearly an Asano order, and hereditary, so it is a Dedekind prime ring.

The following example illustrates some points at which Asano orders can differ from Dedekind prime rings.

4.3.2 Example ([15], page 448): Let R be as in 4.3.1 above. Let $S = R[z]$ be the ring of polynomials in a commuting indeterminate z with coefficients in R. Then (see [32]) S is an Asano order. Now $\frac{S}{zS} \simeq R$, which is not Artinian, so by 1.6.33, S is not a Dedekind prime ring. Thus S is not hereditary, and, by 1.6.30, S is not bounded.

4.3.3 Example ([34], example 7.3): Let F be a field of characteristic zero, and F(y) be the field of rational functions in an indeterminate y. Let x be an indeterminate over $F(y)$ which commutes with the elements of F. Now let $R = F(y)[x]$ be the ring of polynomials in x with coefficients in $F(y)$ subject to the relation $xy - yx = 1$. Clearly R is a partial quotient ring of the ring described in 4.3.1 . Let $S = F + xR$, a hereditary Noetherian domain with unique non-zero proper ideal $A = xR$ (see [34]). Clearly $A = A^2$, so A is not invertible. Hence S is not an Asano order. If $A = aS$ for some $a \in S$, then $aS = A^2 = a^2S$, whence $S = aS$, a contradiction. Thus S is not an i.p.r.i.-ring. Now $(1 - x)S$ is an essential right ideal of S which does not contain A, so S is not bounded. Since $\frac{S}{A}$ is a simple ring, so trivially an i.p.r.i.-ring, this example shows that the assumption of 'boundedness' in theorem 4.2.8 cannot be removed.

4.3.4 <u>Example</u>: Let R be any simple ring and let x, y be commuting indeterminates over R. Let $S = R[x,y]$ be the ring of polynomials in x and y with coefficients in R, subject to the relations $x^2 = y^2 = xy = yx = 0$. Then clearly $W(S) = xS + yS$, and $\frac{S}{W(S)} \cong R$, a simple ring, so S is W-simple. It is straightforward to verify that $W(S)$ is not principal as a right ideal of S, but that every proper homomorphic image of S is an i.p.r.i.-ring. Clearly S is of type (ii)(c) in 4.2.9 . Notice that if we let R be a simple Artinian ring in the above, then S is also Artinian.

We note here that the example given in 3.1.5 is of an Artinian p.r.i.-ring that is not an i.p.l.i.-ring. However, the nilpotent radical of this example is a prime, minimal ideal, and every proper homomorphic image of this example is an i.p.l.i.-ring.

Finally, we note that if R is any ring of the form described in 4.2.9(iii), then R is not an i.p.r.i.-ring (this fact is clear from the proof of 4.2.9), but clearly R has a unique minimal ideal contained in every non-zero ideal of R, and every proper homomorphic image of R is an i.p.r.i.-ring.

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